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EE508: Notes on Options Pricing + Homework

Weiner Processes:

- $z(t)$ is a Weiner process when
  1. $dz = \varepsilon\sqrt{dt}$

     $\varepsilon \sim N(0,1)$ (Gaussian with mean 0 & $\sigma^2 = 1$)
  2. values of dz for two different intervals of time, dt, are independent.

Then: 1. $\overline{dz} = 0$
2. $\text{var}\, dz = dt$

- Weiner Process

$$z(T) - z(0) = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta z_i = \lim_{N\to\infty} \sum_{i=1}^{N} \varepsilon_i \sqrt{\Delta t}$$

Thus: 1. $\overline{z(T) - z(0)} = 0$

2. $$\text{var}\,(z(T) - z(0)) = \overline{(z(T) - z(0))^2}$$
$$= \lim_{N\to\infty} \text{var} \sum_{i=1}^{N} \varepsilon_i\sqrt{\Delta t}$$
$$= \lim_{N\to\infty} N\Delta t = T$$

- Generalized Weiner Process

$$dx = a\,dt + b\,dz$$

a = drift rate ; b^2 = variance rate

- With drift rate only ($b = 0$)

$$dx = a\,dt \Rightarrow \frac{dx}{dt} = a \Rightarrow x = x_0 + at$$

- With variance rate $\neq 0$, x is stochastic

$$dx = a\,dt + b\,\varepsilon\sqrt{dt}$$

Then: 1. $\overline{dx} = a\,dt$

2. $\text{var}\, dx = \overline{(dx)^2} = b^2 dt$

Also: $\overline{x(T) - x(0)} = aT$

$$\text{var}\left[x(T) - x(0)\right] = b^2 T$$

- Ito's lemma

Let x follow an Ito process

$$dx = a(x,t)dt + b(x,t)dz \qquad (+)$$

~~For~~ Any function, $G(x,t)$, then follows the process:

$$dG = \underbrace{\left(\frac{\delta G}{\delta x}a + \frac{\delta G}{\delta t} + \frac{1}{2}b^2\frac{\delta^2 G}{\delta x^2}\right)}_{\hat{a}(x,t)}dt + \underbrace{\frac{\delta G}{\delta x}b}_{\hat{b}(x,t)}dz \qquad (☹)$$

Thus, G also follows the Ito process

$$dG = \hat{a}(x,t)dt + \hat{b}(x,t)dz$$

■ PROOF: $\Delta G(x,t)$ can be written in a Taylor series expansion

$$\Delta G = \frac{\delta G}{\delta x}\Delta x + \frac{\delta G}{\delta t}\Delta t + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}\Delta x^2 \qquad (*)$$
$$+ \frac{\delta^2 G}{\delta x \delta t}\Delta x \Delta t + \frac{1}{2}\frac{\delta^2 G}{\delta t^2}\Delta t^2 + H.O.T.$$

Ito process: $\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$

Thus: $\Delta x^2 = b^2\varepsilon^2\Delta t + (\text{H.O.T. in } \Delta t)$

Substituting into $(*)$ and keep only Δx and Δt terms gives

$$\Delta G = \frac{\delta G}{\delta x}\Delta x + \frac{\delta G}{\delta t}\Delta t + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}b^2\varepsilon^2\Delta t + HOT \qquad (++)$$

As $\Delta t \rightarrow dt$, $\varepsilon^2\Delta t \rightarrow E(\varepsilon^2)\Delta t \rightarrow dt$

Take the limit of $(++)$ and substitute $(+)$ for dx:

$$dG = \frac{\delta G}{\delta x}(a\,dt + b\,dz) + \frac{\delta G}{\delta t}dt + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}b^2 dt$$
$$= \left[\frac{\delta G}{\delta x}a + \frac{\delta G}{\delta t} + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}b^2\right]dt + b\frac{\delta G}{\delta x}dz$$

Ta Da!

- **Finance:**

The most widely used model of stock price behavior:

$$dS = \mu S\,dt + \sigma S\,dz \qquad (☺)$$

where: S = price of stock
μ = expected rate of return
σ = stock volitility
z = Weiner process

- Special case: no risk ($\sigma = 0$)

$$dS = \mu S\,dt \Rightarrow \frac{dS}{dt} = \mu S \Rightarrow S = S_0 e^{\mu t}$$

This is exponential growth of compound interest. S_0 = deposit at time = 0

- • Eq (☺) puts risk in investment by allowing stock to go randomly up or down. If $\mu = 0$, your return is totally random.
  • Note: Both return and volitility effects proportional to S. We're interested in % return & risk. (Obvious!)

- Eq (☺) is an Ito process. Recall

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

or, replacing x with S:

$$dS = a(S,t)\,dt + b(S,t)\,dz$$

Compare with (☺):

$$a(S,t) = \mu S$$

$$b(S,t) = \sigma S$$

- $\frac{dS}{S} = \mu\,dt + \sigma\,dz \Leftarrow$ From (☺)

$$E\left[\frac{dS}{S}\right] = \mu\,dt$$

$$var\left(\frac{dS}{S}\right) = \sigma^2 dt$$

Example: Log of Stock Price

Given: $G = \ln S$

$$dS = \mu S\,dt + \sigma S\,dz$$

Ito's lemma: Motivated by ( ),

$$\frac{\delta G}{\delta S} = \frac{1}{S}, \quad \frac{\delta^2 G}{\delta S^2} = \frac{-1}{S^2}, \quad \frac{\delta G}{\delta t} = 0$$

Substitute into Ito's lemma (Eq ( ))

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dz$$

This is a Generalized Weiner Process (see ( ))
Thus, from ( ),

$$\overline{G(T) - G(0)} = \left(\mu - \frac{\sigma^2}{2}\right)T$$

$$\mathrm{var}\left[G(T) - G(0)\right] = \sigma^2 T$$

and

$$G(T) - G(0)$$

$$= \ln S_T - \ln S_0 \sim G\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$

OR

$$\ln S_T \sim G\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right) \qquad ( )$$

S_T is 'lognormal'

■ Alternate view

$$\frac{dS}{S} = \mu\,dt + \sigma\,dz$$

$$\ln S = \mu t + \sigma z + \text{CONSTANT}$$

$$S_T = S_0 e^{\mu t + \sigma z}$$

$\ln S_T$ is thus Gaussian

- Call Option

Example: You own stock. I will pay you \$5 for the option of buying this stock from you in 2 months for \$140.

X = Strike price = \$140
T = time = 2 months
c = option price

(Note: You are selling a 'call' option. I am buying a call option)

European Option: Option can be excercized only at time T. ⇐ we consider this.

(American Option: Excercize option any time.)

- Question: How much money do I make?

Answer: Depends on value of stock at time T. If value, X, is >145 I make money. If less, I loose and you keep my \$5.

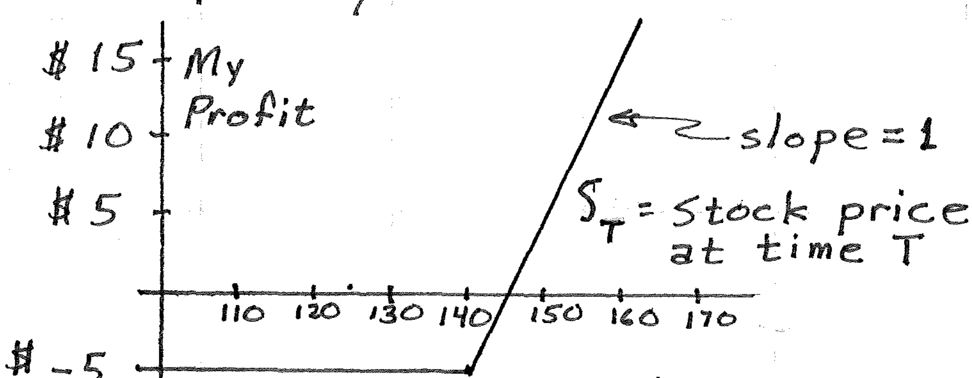


FIG 1.

In General

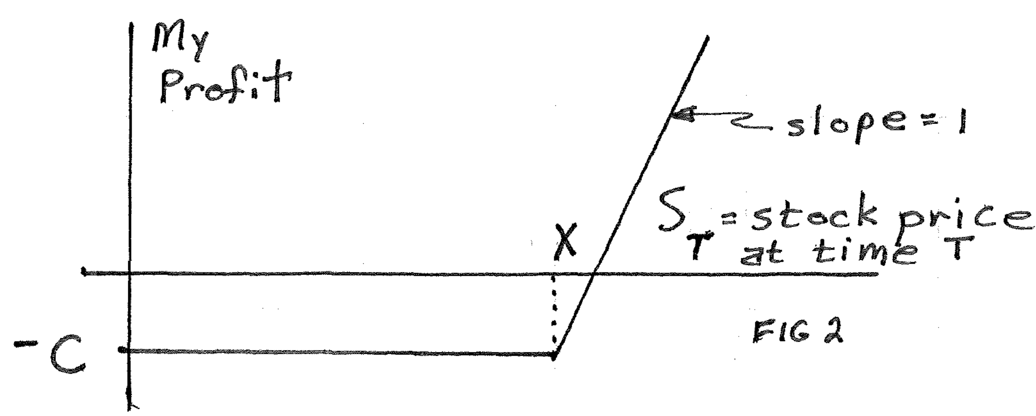
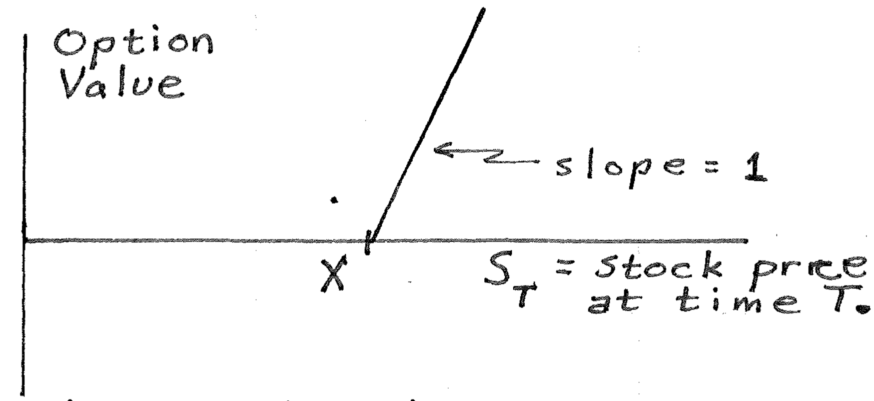


FIG 2

- The 'value' of the option is what I can sell it for at time T



For no interest rate

$$\text{option value} = \max(S_T - X, 0)$$

If r = risk-free interest rate

$$\text{option value} = e^{-rT} \max(S_T - X, 0)$$

(ie, discount at r)

From this, <u>we can find the fair price, c, of the call option!</u>

$$c = E(\text{option value})$$

$$= e^{-rT} E[\max(S_T - X, 0)]$$

~~Note: we know the distribution of S_T:~~

$$\cancel{dS = \mu S\,dt + \sigma S\,dz}$$ ~~← same as ☺~~

~~Ito process with~~ $\cancel{a(S,t) = \mu S}$ $\cancel{b(S,t) = \sigma S}$

Evaluating price, c, of a call option

$$c = e^{-rT} E[\max(S_T - X, 0)]$$

$$= e^{-rT} \int_X^\infty (S_T - X)\, g_{S_T}(S_T)\, dS_T$$

Recall, $g_{S_T}(S_T)$ = pdf of S_T, is lognormal.

Let

$$S_T = e^W \Rightarrow W = \ln S_T$$

Then

$$c = e^{-rT} \int_{\ln X}^\infty (e^W - X)\, g_{S_T}(e^W)\, (e^W dW)$$

From (☺), $e^W g_{S_T}(e^W) = f(W)$ is normal:

$$W = \ln S_T \sim G\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right)$$

Define: $$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} d\xi$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\xi^2/2} d\xi$$

Then, after lots of number crunching:

$$\left.\begin{aligned} c &= S_0 N(d_1) - X e^{-rT} N(d_2) \\ d_1 &= \frac{\ln(S_0/X) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S_0/X) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned}\right\} \text{(B.S.)}$$

The last 3 formulas constitute the Black-Scholes pricing formula for a European ~~American~~ call option. A similar derivation gives the put option price.

- Historical reference
  F. Black & M Scholes "The Pricing of Options and Corporate Liabilities" J. Political Economy, 81 (May-June 1973), pp 637-59

- Technical reference
  John C. Hull Options, Futures & other Derivative Securities, 2nd edition, (Prentice-Hall, 1993).

Homework:

1. Evaluate the Ito process obeyed by

$$S_T = \ln W$$

2. FIGURE 2 shows my profit when I purchase a European call option from you. Sketch the corresponding curve for <u>your</u> profit. (Note: Options are a zero sum game!)

3.(a) Using intuition, what should the cost of a European call option be when:

$$r=0, \quad \sigma = \text{volitility} = 0, \quad S_0 = X = \text{strike price}$$

(b) Confirm your answer by evaluating c using Equations (B.S.)

4. Evaluate the optimal f when your earnings are $P = X$ if $X > 0$ or loss $L = -X$ if $X < 0$ when X is a random variable from some density, $f_X(x)$, with positive mean. Choose your own density.

1. $S_T = \ln W \Rightarrow G = e^{S_T}$

$$dG = \left(\frac{\delta G}{\delta S_T} a + \frac{\delta G}{\delta t} + \frac{1}{2} b^2 \frac{\delta^2 G}{\delta S_T^2} \right) dt + \frac{\delta G}{\delta S_T} b \, dz$$

$$dS_T = \mu S_T \, dt + \sigma S \, dz$$

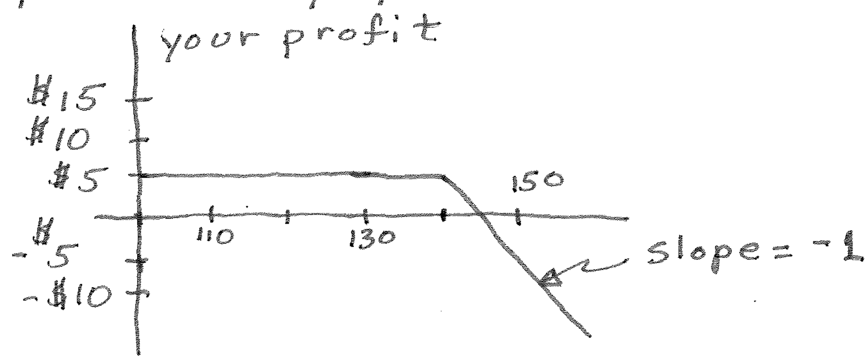
$$\frac{\delta G}{\delta S_T} = e^{S_T} \quad ; \quad \frac{\delta G}{\delta t} = 0 \, , \quad \frac{\delta^2 G}{\delta S_T^2} = e^{S_T}$$

$$a = \mu S_T \quad , \quad b = \sigma S_T$$

$$\therefore dG = \left(e^{S_T} \mu S_T + 0 + \tfrac{1}{2} (\sigma S_T)^2 e^{S_T} \right) dt + e^{S_T} \sigma S_T dz$$

or $dG = S_T e^{S_T} \left[\left(\mu + \tfrac{1}{2} \sigma^2 S_T \right) dt + \sigma dz \right]$

2. Zero Sum Game:
Your profit + my profit = 0



3. (a) The option price should be zero because the price does not change!

(b) $C = S_0 N(d_1) - X \overset{1}{\cancel{e^{-rT}}} N(d_2)$

$$S_0 / X = 1 \Rightarrow d_1 = \frac{(r + \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}} = \infty \quad \text{since } \sigma = 0$$

$$d_2 = d_1 - \sigma\sqrt{T} = \infty$$

$$\therefore C = S_0 N(\infty) - X N(\infty)$$

$$N(\infty) = 1$$

$$\Rightarrow C = S_0 - X = 0 \text{ since } S_0 = X.$$

(1) $P_r[5e^{t}] = \frac{1}{6}$; $P_r[e^{-t}] = \frac{5}{6}$

1. $f_{X}(x;t) = \frac{1}{6}\,\delta(x - 5e^{t}) + \frac{5}{6}\,\delta(x - e^{-t})$

2. $\overline{X(t)} = \frac{1}{6}\cdot 5e^{t} + \frac{5}{6}e^{-t} = \frac{5}{6}(e^{t} + e^{-t}) = \frac{5}{3}\cosh t$

4. $R_{X}(t_1, t_2) = \overline{X(t_1)\,X(t_2)}$

$$= \frac{1}{6} \times 5e^{t_1} \times 5e^{t_2} + \frac{5}{6} \times e^{-t_1} \times e^{-t_2}$$

$$= \frac{25}{6} e^{(t_1+t_2)} + \frac{5}{6} e^{-(t_1+t_2)}$$

$$= \frac{5}{6}\left[5e^{(t_1+t_2)} + e^{-(t_1+t_2)}\right]$$

3. $\overline{X^2(t)} = R_X(t,t) = \frac{5}{6}\left[5e^{t^2} + e^{-t^2}\right]$

$$\text{var } X = \overline{X^2(t)} - \overline{X(t)}^2$$

$$= \frac{25}{6}e^{t^2} + \frac{5}{6}e^{-t^2} + \frac{25}{36}(e^{t} + e^{-t})^2$$

$$= \frac{150}{36}e^{t^2} + \frac{30}{36}e^{-t^2} + \frac{25}{36}(e^{t^2} + e^{-\cancel{2}t^2}) + \frac{50}{36}$$

$$= \frac{25}{18} + \frac{175}{36}e^{t^2} + \frac{55}{36}e^{-t^2}$$

(2) $Y(t) = \sum_n h(t - t_n) = h(t) * \sum_n \delta(t - t_n)$

$= h(t) * X(t)$; $X(t) = \sum_n \delta(t - t_n)$

Recall: $S_X(\omega) = \lambda + 2\pi\lambda^2 \delta(\omega)$

Thus: $S_Y(\omega) = S_X(\omega) \, |H(\omega)|^2$

$= [\lambda + 2\lambda^2\pi\,\delta(\omega)] \, |H(\omega)|^2$

$\int_{-\infty}^{\infty} h(t)\,dt = 0 \Rightarrow H(0) = 0$ and

$S_Y(\omega) = \lambda \, |H(\omega)|^2$

(3) $Y(t) = X(t) + \frac{dX(t)}{dt}$; $R_X(t_1, t_2) = \min(t_1, t_2)$

$Y(t_2) = X(t_2) + \frac{d}{dt_2} X(t_2)$

$$\Rightarrow R_Y(t_1, t_2) = R_X(t_1, t_2) + \frac{d}{dt_2} R_X(t_1, t_2)$$

$$= \min(t_1, t_2) + \frac{d}{dt_2} \min(t_1, t_2)$$

$$= \min(t_1, t_2) + \frac{d}{dt_2} \begin{cases} t_1 & ; t_1 < t_2 \\ t_2 & ; t_1 > t_2 \end{cases}$$

$$= \min(t_1, t_2) + \begin{cases} 0 & ; t_1 < t_2 \\ 1 & ; t_1 > t_2 \end{cases}$$

$$= \min(t_1, t_2) + U(t_1 - t_2)$$

$Y(t_1) = X(t_1) + \frac{d}{dt_1} X(t_1)$

$$R_Y(t_1, t_2) = R_X(t_1, t_2) + \frac{d}{dt_1} R_X(t_1, t_2)$$

$$= [\min(t_1, t_2) + U(t_1 - t_2)]$$

$$+ [U(t_2 - t_1) + \delta(t_1 - t_2)]$$

$$= 1 + \min(t_1, t_2) + \delta(t_1 - t_2)$$

(since $U(t_1 - t_2) + U(t_2 - t_1) = 1$)

(4) $Y[n] - Y[n-1] = X[n]$

$$(1 - z^{-1})\, Y_z(z) = X_z(z)$$

Thus:

$$H(z) = \frac{1}{1 - z^{-1}} = Y_z(z) \big/ X_z(z)$$

$$S_Y(e^{j\Omega}) = S_X(e^{j\Omega})\, |H(e^{j\Omega})|^2$$

$$H(e^{j\Omega}) = \frac{1}{1 - e^{j\Omega}}$$

$$|H(e^{j\Omega})|^2 = \frac{1}{2(1 - \cos\Omega)}$$

$$S_X(e^{j\Omega}) = 9$$

$$\Rightarrow S_Y(e^{j\Omega}) = \frac{9}{2(1 - \cos\Omega)}$$

(5) $X(t)$ is zero mean white Gaussian noise

$$R_X(\tau) = \overline{X^2}\, e^{-\alpha|\tau|}$$

$$E[X(t_1)\, X(t_2)\, X(t_3)] = \overline{X(t_1)}\ \overline{X(t_2)}\ \overline{X(t_3)} = 0 \quad \text{if } t_1 \neq t_2 \neq t_3 \neq t_1$$

$$= \overline{X^2(t_1)}\ \overline{X(t_3)} \quad \text{if } t_1 = t_2 \neq t_3$$

etc.

If $t_1 = t_2 = t_3$

$E[X^3(t_1)] = 3^{\underline{rd}}$ moment of zero mean Gaussian

$= 0$

Thus: $E[X(t_1)\, X(t_2)\, X(t_3)] = 0$

(6) $X(t) = e^{j\Omega t}$; $\Omega \sim f_\Omega(\omega) \leftrightarrow \Phi(t)$

$\Phi(t) = \overline{e^{j\Omega t}}$

$Y(t) = X(t) - \Phi(t)$; $\overline{Y(t)} = 0$

$$R_Y(t_1, t_2) = E\left[(X(t_1) - \Phi(t_1))(X^*(t_2) - \Phi^*(t_2)) \right]$$

$$= E\left[(e^{j\Omega t_1} - \Phi(t_1))(e^{-j\Omega t_2} - \Phi^*(t_2)) \right]$$

$$= \overline{e^{j\Omega(t_1 - t_2)}} - \overline{e^{j\Omega t_1}}\,\Phi^*(t_2) - \Phi(t_1)\,\overline{e^{-j\Omega t_2}} + \Phi(t_1)\Phi^*(t_2)$$

$$= \Phi(t_1 - t_2) - \Phi(t_1)\Phi^*(t_2)$$

<u>NO</u>, NOT WSS.

jump

Review

$X \sim f_X(x)$

X = random variable

$f_X(x)$ = probability density

$P_r[a < x < b] = \int_a^b f_X(x)\,dx \Leftarrow$ If continuous

$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi$ = cumulative distribution

$F_X(\infty) = 1 \quad ; \quad f_X(x) = \frac{d}{dx} F_X(x)$

$E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

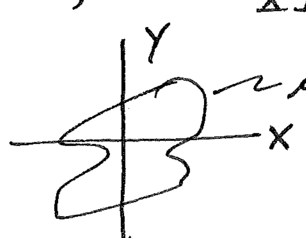
η = mean = $\overline{X}$

$\operatorname{var} X$ = variance = $E[(X - \overline{X})^2]$

$= \overline{X^2} - \overline{X}^2$

Joint R.V.'s

$X, Y \sim f_{XY}(x, y)$



$P_r[(X, Y) \in A]$

$= \int_A f_{XY}(x,y)\,dx\,dy$

$\overline{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,g(x,y)\,dx\,dy$

X and Y are independent

if $f_X(x) f_Y(x) = f_{XY}(x, y)$

Uncorrelated if

$\overline{XY} = \overline{X}\;\overline{Y}$

UNCOR

IND

Explain Why

Marginal Densities

$$f_{X}(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X}(\xi,\eta)\,d\xi\,d\eta$$

$$f_{XY} = \frac{\delta^2}{\delta x \delta y} F_{XY}(x,y)$$

EE505 Check

Do you know:

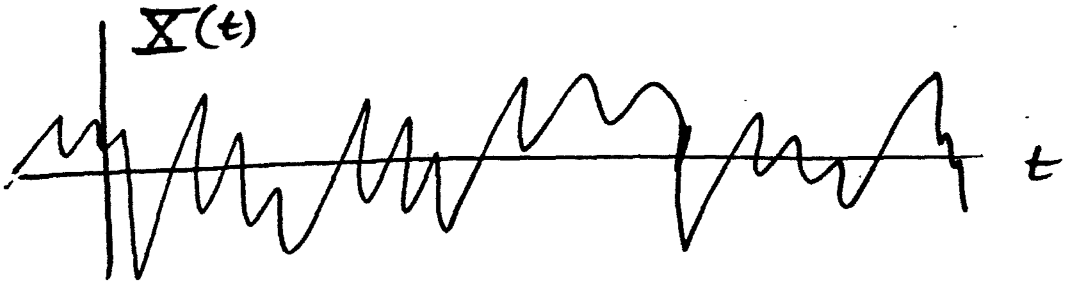
1. Central Limit Theorem?
2. $Z = X + Y$, X & Y are Ind

   $X \sim f_X \quad Y \sim f_Y$

   $f_Z = f_X * f_Y$
3. Characteristic Function
4. $X \sim f_X(x)$

   $Y = X^2 \Rightarrow f_Y = ?$
5. $\frac{A}{x^2+\alpha^2}$ var $= \infty$

STOCHASTIC PROCESSES



Random variable depends on time t.

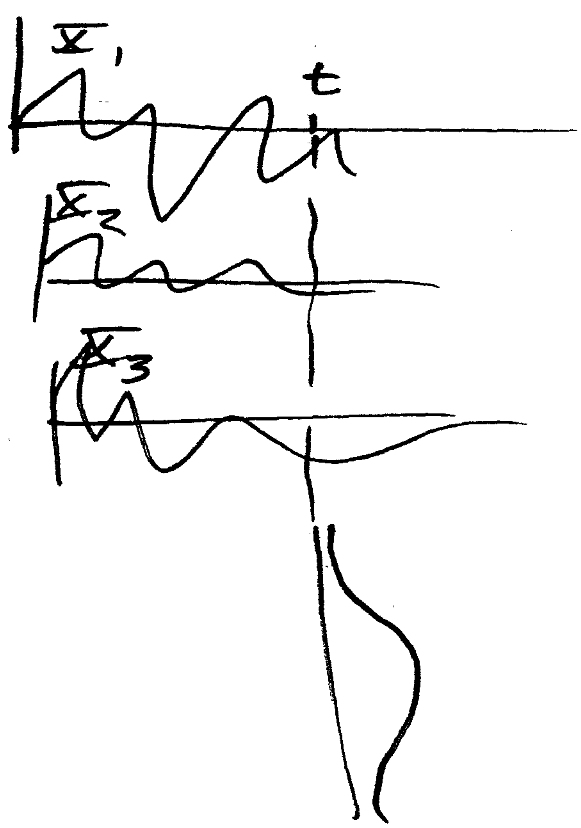
Set t.

~~Define~~

Corresponding R.V. has pdf:

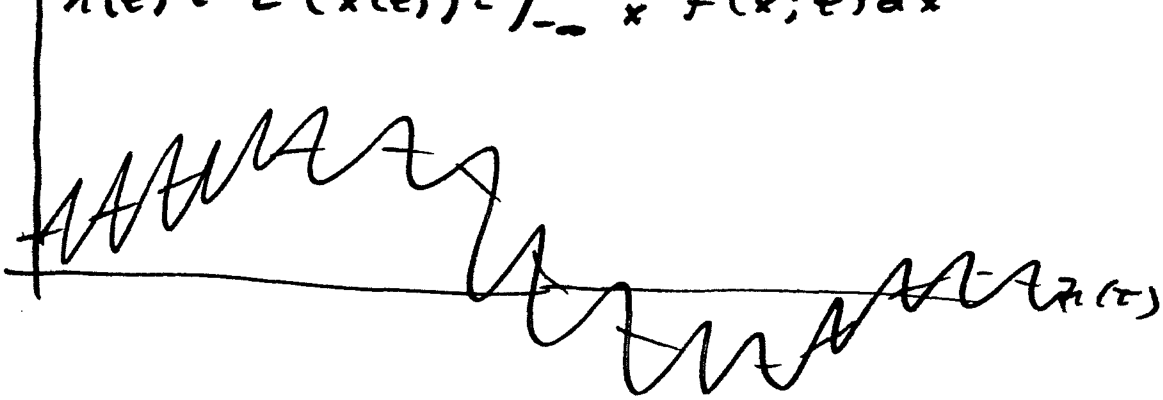
$$f(x;t) = \frac{\delta F(x,t)}{\delta x}$$

<u>Ensemble (rel freq)</u>



Moments:

mean $\eta(t) = E(x(t)) = \int_{-\infty}^{\infty} x\, f(x;t)\,dx$



Autocorrelation

$$R(t_1,t_2) = E[\mathbf{X}(t_1)\,\mathbf{X}^*(t_2)]$$
$$= \iint_{-\infty}^{\infty} x_1 x_2\, f(x_1, x_2; t_1, t_2)\,dx_1\,dx_2$$

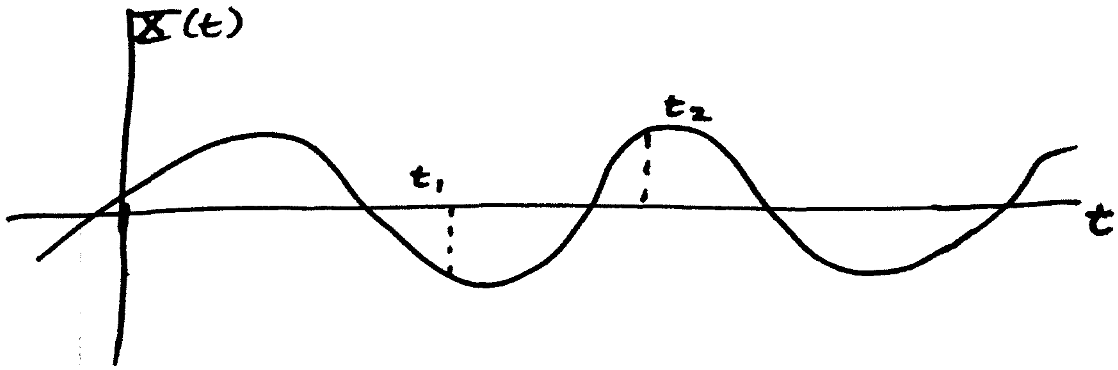
Them: $R(t_1,t_2) = R^*(t_2,t_1)$

AUTO COVARIANCE:

$$C(t_1,t_2) = E[(\mathbf{X}(t_1) - \eta(t_1))(\mathbf{X}(t_2) - \eta(t_2))]$$
$$= R(t_1,t_2) - \eta(t_1)\,\eta(t_2)$$

Note:

$$\sigma^2_{x(t)}(t) = C(t,t)$$
$$= R(t,t) - \eta^2(t)$$



$X(t_1)$ is related (maybe) to $X(t_2)$.

Define:

second order distribution → $F(x_1, x_2; t_1, t_2) = P_r[X(t_1) \le x_1,\ X(t_2) \le x_2]$

$$f(x_1, x_2; t_1\ t_2) = \frac{\delta^2}{\delta x_1 \delta x_2} F(x_1, x_2; t_1, t_2)$$

FIRST ORDER DISTR:

$$f_1(x_1, t_1) = \int_{x_2} f(x_1, x_2; t_1, t_2)\, dx_2$$

Review

$E(A)$ = EXPECTATION

$= \int A\, f_X(x)\, dx$

E is linear

$E(A+B) = E(A) + E(B)$ ← ADD

$E[aA] = a\,E(A)$ ← HOMO

① $R_{\mathbf{X}}(t,t) = \operatorname{var} \mathbf{X}(t)$

② $\cancel{R_{\mathbf{X}}(t_1,t_2)}$

$$\cancel{= \iint_{-\infty}^{\infty} x_1 x_2 f_{\mathbf{X}}(x_1 x_2 ; t_1 t_2)}$$

② $R_{\mathbf{X}}(t_1,t_2) = R_{\mathbf{X}}^{*}(t_2,t_1)$

Ex

GIVEN

$$\eta_X(t) = 3 \qquad R_X(t_1, t_2) = 9 + 4e^{-\frac{|t_1 - t_2|}{5}}$$

$$Z = X(5)$$

$$\overline{Z} = \overline{X(5)} = \eta(5) = 3$$

$$\overline{Z^2} = R_X(5,5) = 13$$

$$\text{var } Z = 13 - 9 = 4$$

$$W = X(8)$$

$$\overline{WZ} = E[X(5)X(8)]$$

$$= R_X(5,8) = 9 + 4e^{-3/5} = 11.195$$

EE508 Midterm Examination #2

Winter 1997

Instructions.

- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed two notebook sized sheet of notes & a calculator.
- Each problem is worth 20 points.

1 Problem

A common model for the autocovariance of certain stochastic processes is

$$C_X(\tau) = \mathrm{var}(X)\, \mathrm{e}^{-\lambda|\tau|} \cos(\omega_0 \tau)$$

Does the integral

$$\frac{1}{2T} \int_{-T}^{T} X(t)dt$$

converge to the (non-zero) mean of $X(t)$? In other words, is $X(t)$ ergodic in the mean?

2 Problem

A securities price, S, follows the familiar Ito process

$$dS = \mu S dt + \sigma S dz.$$

Describe the Ito process for S^2.

3 Problem

Consider the first order difference equation

$$Y[n-1] + 2Y[n] = 4X[n]$$

What is the power spectral, $S_Y(e^{j\omega})$, when the input is discrete white noise with power spectral density of $S_X(e^{j\omega}) = q$?

4 Problem

A stochastic process, $X(t)$, has an autocorrelation of

$$R_X(t_1, t_2) = e^{-|t_1|} + e^{-|t_2|}$$

Let $Y(\omega)$ be the Fourier transform of $X(t)$.

$$Y(\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

Evaluate the autocorrelation, $R_Y(\omega_1, \omega_2)$ of the stochastic process $Y(\omega)$.

5 Problem

Let $X(t)$ denote a Wiener process with parameter α. Define the stochastic process

$$Z(t) = X(t) - X(t - \tau)$$

where τ is a given positive number. Determine the first order probability density function of $Z(t)$ for all positive values of t.

Scratch Paper #1

Scratch Paper #2

Scratch Paper #3

Scratch Paper #4

Robert J Marks II

EE508 - HOMEWORK #5

CHAPTER 11

11-1

(a) If $\underset{\sim}{x}(t)$ is a Poisson process as in Fig. 10-3a, then for a fixed t, $\underset{\sim}{x}(t)$ is a Poisson RV with parameter λt. Hence [see (5-79)] its characteristic function equals $\exp\{\lambda t(e^{j\omega}-1)\}$.

(b) If $\underset{\sim}{x}(t)$ is a Wiener process then $f(x,t)$ is $N(0,\sqrt{\alpha t})$. Hence [see (5-65)] its first order characteristic function equals $\exp\{-\alpha t\omega^2/2\}$.

11-2 For large t, $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ can be approximated by two independent Wiener processes as in (11-4):

$$f_x(x,t) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-x^2/2\alpha t} \qquad f_y(y,t) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-y^2/2\alpha t}$$

Hence, $\underset{\sim}{z}(t)$ has a Rayleigh density [see (6-50)]. [Note. Exactly, $\underset{\sim}{z}(t)$ is a discrete-type RV taking the values $s\sqrt{m^2+n^2}$ where m and n are integers]. The product $f_z(z,t)dz$ equals approximately the probability that $\underset{\sim}{z}(t)$ is between z and $z+dz$ provided that $dz \gg T$.

---

file:///H|/ciaweb/Marks-Stuff/Classes/508hw5.htm

11-3 The voltage $\underline{v}(t)$ is the output of a system with input $\underline{n}_e(t)$ and system function

$$H_1(s) = \frac{1}{LCs^2 + RCs + 1}$$

Hence,

$$S_v(\omega) = S_{n_e}(\omega)|H_1(j\omega)|^2 = \frac{2kTR}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

Furthermore,

$$Z_{ab}(s) = \frac{R + Ls}{LCs^2 + RCs + 1} \qquad \underline{\mathrm{Re}}\ Z_{ab}(j\omega) = \frac{R}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

in agreement with (11-27).

The current $\underline{i}(t)$ is the output of a system with input $\underline{n}_e(t)$ and system function

$$H_2(s) = \frac{1}{R + Ls}$$

Hence,

$$S_i(\omega) = S_{n_e}(\omega)|H_2(j\omega)|^2 = \frac{2kTR}{R^2 + \omega^2 L^2}$$

Furthermore (short circuit admittance)

$$Y_{ab}(s) = \frac{1}{R + LS} \qquad \underline{\mathrm{Re}}Y_{ab}(j\omega) = \frac{2kTR}{R^2 + L^2\omega^2}$$

in agreement with (11-30).

11-4 The equation $m\underline{x}''(t) + f\underline{x}'(t) = \underline{F}(t)$ specifies a system with

$$H(s) = \frac{1}{ms^2 + fs} \qquad h(t) = \frac{1}{f}(1 - e^{-ft/m})U(t)$$

and (10-90) yields

$$E\{\underline{x}^2(t)\} = \frac{2kTf}{f^2}\int_0^t (1 - e^{-2\alpha\tau})^2 d\tau \qquad \alpha = \frac{f}{2m}$$

---

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11-6 If $x(t) = w(t^2)$ then [see (11-22)]

$$R_x(t_1,t_2) = E\{w(t_1^2)w(t_2^2)\} = \alpha t_1^2$$

If $y(t) = w^2(t)$ then [see (7-36)]

$$R_y(t_1,t_2) = E\{w^2(t_1)w^2(t_2)\}$$

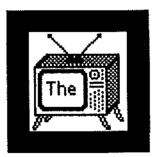
$$= Ew^2(t_1)E\{w^2(t_2)\} + 2\,E^2\{w(t_1)w(t_2)\} = \alpha^2 t_1 t_2 + 2\alpha^2 t_1^2$$

109

11-7 From (11-61):

$$\eta_s = 3\int_0^{10} 2\,dt = 60 \qquad \sigma_s^2 = 3\int_0^{10} 4dt = 120 \qquad E\{s^2\} = 3720$$

$s(7) = 0$ if there are no points in the interval (7-10, 7). The number of points in this interval is a Poission RV with parameter $10\lambda = 30$. Hence, $P\{s(7) = 0\} = e^{-30}$.



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11-14

The process

$$y_N(t) = x(t+\tau) - \sum_{n=-N}^{N} x(t+nT)\,\frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}$$

is the output of a system with input $x(t)$ and system function

$$H_N(\omega) = e^{j\omega\tau} - \sum_{n=-N}^{N} \frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}\,e^{jnT\omega}$$

Furthermore, $\varepsilon_N(\tau) = y_N(0)$, hence [see (10-139)]

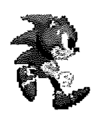
$$E\{\varepsilon_N^2(\tau)\} = E\{y^2(0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,|H_N(\omega)|^2 d\omega \qquad (i)$$

The function $H_N(\omega)$ is the truncation error in the Fourier series expansion of $e^{j\omega\tau}$ in the interval $(-\sigma,\sigma)$. Hence, for $N > N_0$

$$|H_N(\omega)| < \varepsilon \qquad |\omega| < \sigma$$

From this and (i) it follows that, if $S(\omega) = 0$ for $|\omega| < \sigma$, then

$$E\{\varepsilon_N^2(\tau)\} = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\,|H_N(\omega)|^2 d\omega < \varepsilon\, R(0) \qquad N > N_0$$



CHAPTER 13

13-1 $\underset{\sim}{x}(t) = 10 + \underset{\sim}{\nu}(t)$ $\quad R_\nu(\tau) = 2\,\delta(\tau)$ $\quad E\{\underset{\sim}{\nu}(t)\} = 0$

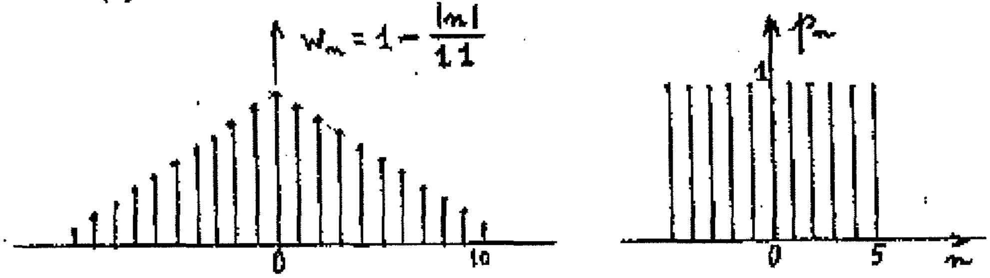
$$E\{\underset{=}{n}_T\} = E\{\underset{\sim}{x}(t)\} = 10 \qquad C_x(\tau) = 2\delta(\tau)$$

From (13-5)

$$\sigma^2_{n_T} = \frac{1}{2T}\int_{-T}^{T} C_x(\tau)\left(1 - \frac{|\tau|}{2T}\right) d\tau = \frac{1}{2T}\int_{-T}^{T} 2\delta(\tau)\ \left(1 - \frac{|\tau|}{2T}\right) d\tau = \frac{1}{T}$$

13-12 (a) It follows from the convolution theorem for Fourier series

(b)



With p_n as above, $w_n = \frac{1}{11} p_n * p_n$

$$P(\omega) = \sum_{n=-5}^{5} e^{-jnT\omega} = \frac{\sin 5.5\omega T}{\sin 0.5\omega T} \qquad W(\omega) = \frac{1}{11} P^2(\omega)$$

EE508 Final Examination

Winter 1994

March 16, 1994

Instructions.

- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed two notebook sized sheet of notes & a calculator.
- Each problem is worth 20 points.

You will find the following Fourier transform pairs useful

$$e^{|\tau|} \Leftrightarrow \frac{2}{1+\omega^2}.$$

$$\text{If } x(t) \Leftrightarrow X(\omega), \text{ then } x(at) \Leftrightarrow \frac{1}{|a|}X(a\omega)$$

1 Problem

The autocorrelation of a zero mean stochastic process is

$$R_X(\tau) = \mathrm{e}^{-\alpha|\tau|}$$

Design a linear time invariant filter using adders, differentiators and amplifiers/attenuators so that, when $X(t)$ is fed into the filter, the output, $Y(t)$, is white.

2 Problem

A common model for the autocovariance of certain stochastic processes is

$$C_X(\tau) = \mathrm{var}(X)\mathrm{e}^{-\lambda|\tau|}\cos(\omega_0\tau)$$

Does the integral

$$\frac{1}{2T}\int -T^T X(t)dt$$

converge to the (non-zero) mean of $X(t)$? In other words, is $X(t)$ ergodic in the mean?

3 Problem

Consider the first order difference equation

$$Y[n-1] + 2Y[n] = 4X[n]$$

What is the power spectral, $S_Y(e^{j\omega})$, when the input is discrete white noise with power spectral density of $S_X(e^{j\omega}) = q$?

4 Problem

A stochastic process, $X(t)$, has an autocorrelation of

$$R_X(t_1, t_2) = \mathrm{e}(-|t_1|) + \mathrm{e}(-|t_2|)$$

Let $Y(\omega)$ be the Fourier transform of $X(t)$.

$$Y(\omega) = \int_{-\infty}^{\infty} X(t)\mathrm{e}^{-j\omega t} dt$$

Evaluate the autocorrelation, $R_Y(\omega_1, \omega_2)$ of the stochastic process $Y(\omega)$.

5 Problem

Let $X[n]$ denote a real discrete stationary Gaussian stochastic process with mean η_X and variance $\mathrm{var}(X)$. Let K denote a binomial random variable corresponding to N trials with a probability of success p. Let K and X be independent. Consider the sum

$$S = \sum_{k=0}^{K} X^2[n]$$

Evaluate the expected value of the random variable S.

Scratch Paper #1

Scratch Paper #2

Scratch Paper #3

Scratch Paper #4

EE508 Examination

Robert J. Marks II

Please write your name on the upper right hand corner.

This examination is closed book and closed notes. Calculators are allowed but will probably not be needed. Each student is allowed one $8\frac{1}{2}$ by 11 sheet of notes for the test.

All work will be done in this test booklet.

Each problem is worth 20 points.

1. **Fundamentals**. Circle the correst answer. Correct answers are +4, incorrest answers are -2 and no answer gives a zero score.

   (a) The expectation of a Fourier transform is the same as the Fourier transform of the expectation.
   TRUE FALSE

   (b) A random variable's second moment is always finite.
   TRUE FALSE

   (c)
   $$\overline{X^2} \geq \overline{X}^2$$
   TRUE FALSE

   (d)
   $$F_X(x) \geq F_X(x-1)$$
   TRUE FALSE

   (e) $\overline{X}$ is always real.
   TRUE FALSE

   (f) The characteristic function for the degenerate case of a deterministic "random variable" is always a complex exponential.
   TRUE FALSE

2. A zero mean Gaussian stochastic process, $X(t)$, has an autocovariance of

$$C_X(t_1, t_2) = \frac{1}{2\pi} \min(|t_1|, |t_2|)$$

(a) Find the three dimensional charateristic function, $\Phi_X(\omega_1, \omega_2, \omega_3)$ for the random variables $X(-1)$, $X(0)$ and $X(1)$.

(b) Find the corresponding probability density function.[1]

[1]Recall that $\exp(-\pi t^2)$ has a Fourier transform of $\exp\left(-\pi (\frac{\omega}{2\pi})^2\right)$.

3. The autocorrelation of $X(t)$ is

$$R_X(t_1, t_2) = \overline{X^2}\cos(t_1 t_2)$$

Given the initial condition $Y(0) = 0$, find $R_Y(t_1, t_2)$ when

$$\frac{d}{dt}Y(t) = X(t)$$

Express your answer in terms of the *sine integral*

$$\mathrm{Si}(t) = \int_0^t \frac{\sin(\tau)}{\tau} d\tau$$

4. A Gaussian stochastic process, $X(t)$, and its Fourier transform, $\chi(\omega)$, are described as follows.

$$\overline{X(t)} = \mathrm{e}^{-\pi t^2}$$

$$R_X(t_1, t_2) = \mathrm{e}^{-\min(|t_1|,|t_2|)}$$

$$\chi(\omega) = \int_{-\infty}^{\infty} X(t)\mathrm{e}^{-j\omega t} dt$$

Find $\overline{\chi(\omega)}$.

5. A *whitening filter* is a linear time invariant filter that transforms an input, $X(t)$, into a white stochastic process with autocorrelation, $R_Y(\tau) = q\delta(\tau)$. Specify the frequency response, $H(\omega)$, of a whitening filter when the autocorrelation of the input is

$$R_X(\tau) = \mathrm{e}^{-\pi\tau^2}.$$

6. A digital filter has a transfer function equal to

$$H(z) = \frac{1}{z - \frac{1}{2}}$$

The input to the filter is discrete white noise with autocorrelation

$$R_X[n] = q\ \delta[n]$$

Let the filter output be $Y[n]$. Find the power spectral density, $S_Y(e^{j\omega})$, of the output.

scratch paper

2. scratch paper

5. scratch paper

Solutions

EE508 Examination

Robert J. Marks II

- Please write your name on the upper right hand corner.
- This examination is closed book and closed notes. Calculators are allowed but will probably not be needed. Each student is allowed two $8\frac{1}{2}$ by 11 sheets of notes for the test.
- All work will be done in this test booklet.
- Each problem is worth 20 points.

1. The process $w(t)$ is a Wiener process with parameter α. Form the stochastic process

$$z(t) = |w(t)|$$

(a) Evaluate the mean of $z(t)$.

(b) Write a Fourier integral equation for the first order characteristic function of $z(t)$. You need not solve the equation.

(a) $w(t) \sim \frac{1}{\sqrt{2\pi\alpha t}} e^{-w^2/2\alpha t}$

$\Rightarrow z(t) \sim \frac{2}{\sqrt{2\pi\alpha t}} e^{-w^2/2\alpha t} \quad U(w)$

$\overline{z(t)} = \frac{2}{\sqrt{2\pi\alpha t}} \int_0^\infty w\, e^{-w^2/2\alpha t}\, dw$

$= \frac{\cancel{2}}{\sqrt{2\pi\alpha t}} \quad \frac{2\alpha t}{-\cancel{2}} \; e^{-w^2/2\alpha t}\Big|_0^\infty$

$= \sqrt{\frac{2\alpha t}{\pi}}$

(b) $\Phi_z(\omega) = \int_{-\infty}^{\infty} f_z(z)\, e^{-j\omega z}\, dz$

$= \frac{2}{\sqrt{2\pi\alpha t}} \int_0^\infty e^{-\frac{w^2}{2\alpha t}}\, e^{-j\omega w}\, dw$

2. Define the (nonstationary) stochastic process

$$X(t) = A\cos(2\omega t - \Theta)\ \mathrm{e}^{-t}\ U(t)$$

where

- A is a zero mean Gaussian random variable with variance σ^2,
- Θ is a uniform random variable, (from 0 to 2π)
- ω is a known angular frequency and
- $U(t)$ is the unit step function.

The two random variables are statistically independent of each other. Is $X(t)$ ergodic in the mean?[1]

FIRST, NOTE $\overline{X(t)} = \overline{A}\ \overline{\cos(2\omega t - \Theta)}\ e^{-t}\ U(t)$

since $\overline{\cos(2\omega t - \Theta)} = \frac{1}{2\pi}\int_0^{2\pi} \cos(2\omega t - \theta)\,d\theta = 0$

$\Rightarrow \overline{X(t)} = 0$

Time average:

$$\langle X(t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\cos(2\omega t - \Theta)\,e^{-t}\,U(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\underbrace{\int_{0}^{T} A\cos(2\omega t - \Theta)\,e^{-t}\,dt}_{\text{Finite as } T\to\infty\ \forall\ \Theta\ \&\ A}$$

$= 0 \Leftarrow$ Deterministic!

$\therefore \operatorname{var}\langle X(t)\rangle = 0$

Yes! Ergodic.

(Could check $\operatorname{var}\langle X(t)\rangle$. Would find it's zero.)

[1] This is not a trick question. There are nonstationary processes that are ergodic in the mean. Is this one of them?

3. A stochastic signal, $X(t)$, obeys an Ito process. Is $X(t)$ a Gaussian process? Explain your answer.

No! Of course not!

The simple finance equation:

$$dS = \mu S\,dt + \sigma S\,dW$$

has a solution that is lognormal $\neq$ Gaussian

$S = X$

5. Define the stochastic process

$$X(t) = e^{j\Omega t}$$

where Ω is a discrete random variable with probability density function

$$f_\Omega(\omega) = \sum_{n=-\infty}^{\infty} p_n\, \delta(\omega - n)$$

where the p_n's are probabilities. Is $X(t)$ a cyclostationary process? Justify your response.

① $\overline{X(t)} = E[e^{j\Omega t}] = \sum_{n=-\infty}^{\infty} p_n \int_{-\infty}^{\infty} e^{j\omega t}\,\delta(\omega - n)\,d\omega$

$= \sum_{n=-\infty}^{\infty} p_n\, e^{jnt} \Leftarrow$ Fourier Series with period $T = 2\pi$.

$\therefore \overline{X(t)}$ is periodic.

② $R_X(t;\tau) = E[X(t)X^*(\tau)]$

$= E[e^{j\Omega t} e^{-j\Omega \tau}] = E[e^{j\Omega(t-\tau)}]$

$R_X(t;\tau) = R_X(t-\tau) \Leftarrow$ This is the autocorrelation we might expect from a stationary process. ~~Since all station~~

Certainly, $R_X(t;\tau) = R_X(t+mT;\ \tau+mT)$

$= R_X(t-\tau)$

Yes $\Rightarrow$ cyclostationary

4. Care must be taken in directly performing a Fourier transform of a stochastic process. Let $X(t)$ be stationary in the wide sense with mean $\mu_X \neq 0$ and autocovariance $C_X(\tau)$.

   - Evaluate the expected value of the Fourier transform of $X(t)$.
   - What is the expected value of the Fourier transform of $X(t)$ at $\omega = 0$?[2]

- $\mathcal{X}(\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$

  $\overline{\mathcal{X}(\omega)} = \overline{X} \int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi \overline{X} \, \delta(\omega) = 2\pi \mu_X \delta(\omega)$

- $\overline{\mathcal{X}(0)} = 2\pi \mu_X \delta(0) = \infty$

[2]The Fourier transform is
$$\int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

EE505 Final Examination

Robert J. Marks II

December 12, 1997

The examination is closed book and closed notes. No calculators are allowed. Each student is allowed three sheets of notes. All problems are weighted equally. Work must be done in ink.

All work will be done in a test booklet. No scratch paper is needed.

"Test everything. Hold on to the good." 1 Thessalonians 5:21 (English-NIV)

1. The events A and B are independent. Are the events $\bar{A}$ and $\bar{B}$ always independent? If yes, explain. If not, provide a counter example.

Yes, $\bar{A}$ is ind of $\bar{B}$.

Proof: $P_B = P_{A\cdot B + \bar{A}B}$; $A \cdot B$ is M.E. of $\bar{A}B$

$P_B = P_{AB} + P_{\bar{A}B}$

$$\Rightarrow P_{\bar{A}B} = P_B - P_{AB} \underset{\substack{\uparrow \\ A \text{ IND } B}}{=} P_B - P_B P_A = P_B(1 - P_A)$$

$$= P_B P_{\bar{A}} \Leftarrow B \text{ ind of } \bar{A}$$

More:

$$P_{\bar{A}} = P_{\bar{A}\bar{B}} + P_{B\bar{A}} \underset{\text{SINCE}}{\Rightarrow} \bar{A}\bar{B} \text{ M.E. of } B\bar{A}$$

$$P_{\bar{A}\bar{B}} = -\underbrace{P_{B\bar{A}}}_{\text{shown ind}} + P_{\bar{A}} = -P_B P_{\bar{A}} + P_{\bar{A}}$$

$$= P_{\bar{A}}(1 - P_B)$$

$$= P_{\bar{A}} P_{\bar{B}}$$

QED

cannot show $P(\overline{A+B}) = P(\bar{A}\bar{B})$ -5

2. Let X and Y be independent random variables and let $Z = X + Y$. Prove or disprove the following general propositions.

(a) $\bar{Z} = \bar{X} + \bar{Y}$ yes

(b) $\overline{Z^2} = \bar{X}^2 + \bar{Y}^2$ No!

(c) $\mathrm{var}(Z) = \mathrm{var}(X) + \mathrm{var}(Y)$.

(d) $\mathrm{var}(aZ) = a^2\, \mathrm{var}Z$. yes

5 (a) $\bar{Z} = \overline{X+Y} = \bar{X} + \bar{Y}$

5 (b) $\overline{Z^2} = \overline{(X+Y)^2} = \overline{X^2} + \overline{Y^2} + 2\bar{X}\bar{Y}$

$\neq \bar{X}^2 + \bar{Y}^2$

5 (c) $\Psi_Z(\omega) = \Psi_X(\omega) + \Psi_Y(\omega)$

$\Rightarrow \Psi_Z''(0) = \Psi_X''(0) + \Psi_Y''(\omega)$ THIS FOLLOWS

5 (d) $\mathrm{var}\, aZ = E\left[\left(aZ - \overline{aZ}\right)^2\right]$

$= E\left[\left(aZ - a\bar{Z}\right)^2\right]$

$= E\left[a^2\left(Z - \bar{Z}\right)^2\right]$

$= a^2 E\left[\left(Z - \bar{Z}\right)^2\right]$

$= a^2\, \mathrm{var}\, Z$

3.

$$Y = \frac{1}{N}\sum_{k=1}^{N} X_k^2$$

where the X_k's are i.i.d. random variables with probability density function

$$f_X(x) = e^{-x}U(x)$$

Estimate the probability density function for the random variable Y when N is large.[1]

Let $Z_k = X_k \Rightarrow \overline{X_k^2} = \int_0^\infty x^2 e^{-x}dx = 2! = 2 = \overline{Z}_k$

$\overline{Z_k^2} = \overline{X_k^4} = \int_0^\infty x^4 e^{-x}dx = 4! = 24$ 9

$\Rightarrow \text{var } Z_k = 24 - 4 = 20$

Thus $\overline{Y} = \frac{1}{N}\sum_{k=1}^{N} \overline{Z}_k = \frac{2}{N}\cdot N = 2$

$\text{var } Y = \frac{1}{N^2} \text{ var } \sum_{k=1}^{N} \text{var}(Z_k) = \frac{1}{N^2}\cdot 20 \cdot N = \frac{20}{N}$ 9

By CLT

$\Rightarrow Y \sim n\left(2, \sqrt{\frac{20}{N}}\right)$ 2

Var (9) = 9 $E[X^2] = 6$

E (9) = 9 $\rightarrow$ Var$[X^2] = 6$

[1] Recall

$$n! = \Gamma(n+1) = \int_0^\infty y^n e^{-y}dy$$

4. A total of N i.i.d. Bernoulli trials with probability of success p are performed. The outcome of trial m, the random variable X_m, is set to one if there is a success and zero otherwise. We form the sum

$$Y = \sum_{m=1}^{N} X_m.$$

Evaluate the exact probability density function for the random variable Y.

Y is binomial. (Y = # successes in n Bernoulli trials)

$$f_Y(y) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \delta(y-k)$$

min 5

for Binomial 10

delta fn 5

→ CLT ?

→ try to find 2q-10

→ convolution ans 12

5. The Weibull random variable Y with positive parameters A and B is

$$F_Y(y) = \left[1 - \exp\left(-\left(\frac{y}{A}\right)^B\right]U(y).$$

Let X be a uniform random variable on the interval $(0,1)$. Given A and B, find a random variable transformation, $Y = g(X)$, to produce a Weibull random variable.

$$X = 1 - e^{-\left(\frac{Y}{A}\right)^B}$$

$$e^{-\left(\frac{Y}{A}\right)^B} = 1 - X$$

$$\left(\frac{Y}{A}\right)^B = -\ln(1-X)$$

$$Y = A\left[-\ln(1-X)\right]^{\frac{1}{B}} = g(X)$$

$$X = 1 - e^{\left(-\frac{y}{A}\right)^B}$$

$$e^{\left(-\frac{y}{A}\right)^B} = 1 - X$$

$$\left(-\frac{y}{A}\right)^B = \ln(1-X)$$

$$-\frac{y}{A} = \left[\ln(1-X)\right]^{\frac{1}{B}}$$

$$y = -A\left[\ln(1-X)\right]^{\frac{1}{B}}$$ for B odd only

min 5

setup 10

~~error~~ 3

$F_X(x) = ~~1/4~~ X$ = 14

error -3

recheck $f_Y(y)$ → no $\delta(y)$

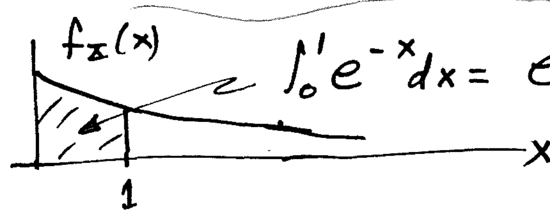
6. A random variable has a probability density function of

$$f_X(x) = e^{-x}U(x)$$

We take i.i.d. samples from this distribution until we get a number between zero and one - and then stop. Call this last random variable Y. Evaluate the probability density function of Y.

min 5

$f_Y(x) = f_X(x \mid 0 \le X \le 1\}$ 16

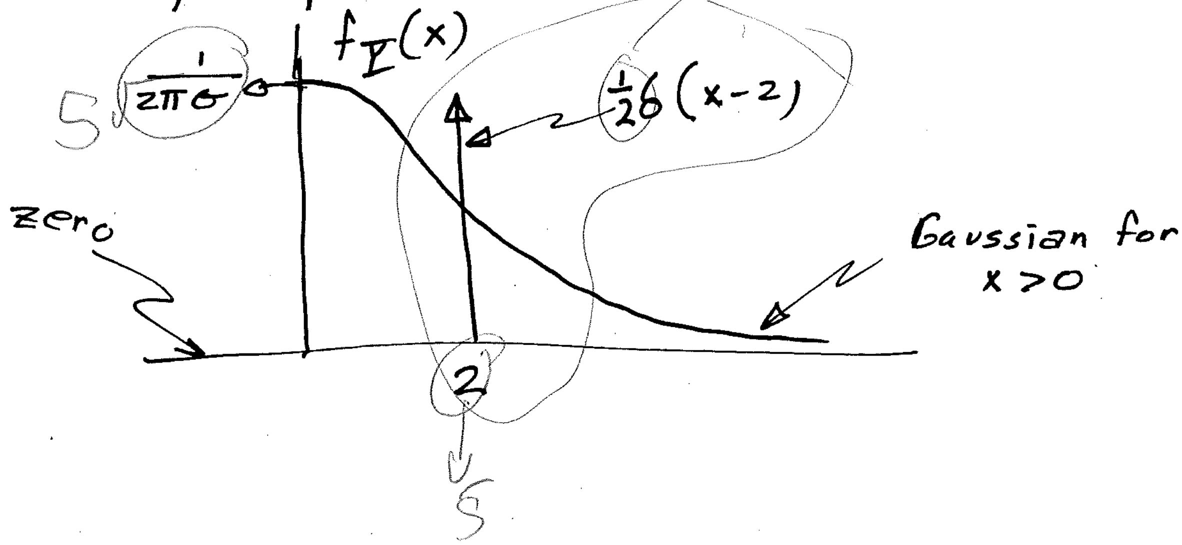


$\int_0^1 e^{-x}dx = e^{-x}\Big|_1^0 = 1 - e^{-1}$ 15

$$f_Y(x) = \begin{cases} \dfrac{e^{-x}}{1-e^{-1}} ; & 0 \le x \le 1 \\ 0 ; & else \end{cases}$$ 20

7. Let X be a zero mean normal random variable with variance σ^2. Let $Y = X$ when X is positive and let $Y = 2$ otherwise. Evaluate and sketch the probability density function for Y.

By inspection:

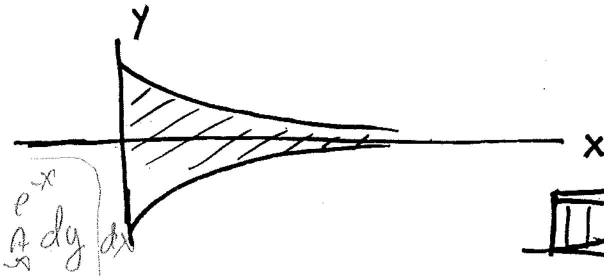


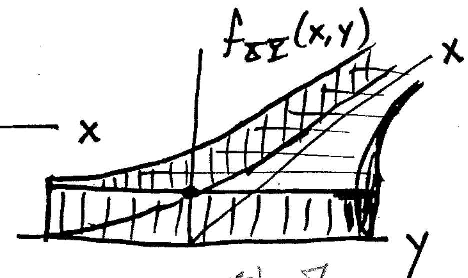
3. A joint probability density function is defined by

$$f_{XY}(x,y) = \begin{cases} A & ; |y| \le e^{-x} \text{ and } x \ge 0 \\ 0 & : \text{otherwise} \end{cases}$$

(a) Evaluate A.

(b) Evaluate the marginal distribution, $f_Y(y)$.





$\int \int_{-e^{-x}}^{e^{-x}} A \, dy \, dx$

(a) Area $= 2A\int_0^\infty e^{-x} dx = 2A \Rightarrow A = \frac{1}{2}$ $\Rightarrow 7$

(b) $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$ $\Rightarrow 3$

$= \int_{y=e^{-x}}^{e^{x}} (\frac{1}{2})\,dy = e^{-y}$; $y > 0$

$\therefore f_Y(y) = e^{-y}U(y)$ $\Rightarrow 10$

$f_Y(y) = \int_0^{-\ln y} A\,dx$, $1>y>0$ (5)

$= \int_0^{-\ln(-y)} A\,dx$; $-1<y<0$ (5)

$= \int_0^{-\ln|y|} A\,dx$

$= -\frac{1}{2}\ln|y|$, $|y|<1$

$= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^{e^{-y}} f(x,y)\,dx$

$= \frac{1}{2} e^{-y}$

$y = e^{-x}$

$dy = -e^{-x}dx$

$\int \ln y \, dy$

9. A die is rolled. The stochastic process, $X(t)$, is set to $5\,\mathrm{e}^{-t}$ when the die shows six dots. Otherwise, the stochastic process is set equal to $-\mathrm{e}^{-t}$.

(a) Evaluate the expected value of $X(t)$.

(b) Evaluate the autocorrelation, $R_X(t_1, t_2)$.

$$(a)\ \overline{X(t)} = \overbrace{\tfrac{1}{6}\, 5e^{-t}}^{3} + \overbrace{\tfrac{5}{6} \cdot (-e^{-t})}^{3}$$

$$= \tfrac{5}{6}(e^{-t} - e^{-t}) = 0$$

8

$$(b)\ R(t_1, t_2) = \overline{X(t_1)\, X(t_2)}$$

12

$$= \underbrace{\tfrac{1}{6}\left[5e^{-t_1} \times 5e^{-t_2}\right]}_{5} + \underbrace{\tfrac{5}{6}(-e^{-t_1})(-e^{-t_2})}_{5}$$

$$= \tfrac{25}{6} e^{-(t_1+t_2)} + \tfrac{5}{6} e^{-(t_1+t_2)}$$

5 ~~4~~

$$= \tfrac{30}{6} e^{-(t_1+t_2)}$$

$$= \tfrac{1}{36}\left[25 e^{-(t_1+t_2)}\right] + \tfrac{5}{36}\left[5e^{-(t_1+t_2)}\right] + \tfrac{5}{36}\left[-5e^{-(t_1+t_2)}\right]$$

$$+ \tfrac{25}{36}\left[e^{-(t_1+t_2)}\right]$$

$$\int_0^1 \ln x\, dx = x\ln x - \int \frac{x}{x}\, dx$$

$$= (x \ln x - x)\Big|_0^1$$

$$= (0 - 1 - \tfrac{\ln 0}{\frac{1}{0}} - 0)$$

$= -1$

$\ln 0 = -\infty$

$$\int \underbrace{\ln x}_{u}\ \underbrace{dx}_{dv}$$

10. The power spectral density of a stochastic process is

$$S_X(\omega) = 3\, e^{-2|\omega|}.$$

What *percentage* of the power is above the frequency $\omega = 1$ radian per second?

Total power: $2 \times \frac{1}{2\pi} \int_0^\infty S_X(\omega)\,d\omega = 2 \times \frac{1}{2\pi} \times \frac{3}{2} = \frac{3}{2\pi}$

Power for $\omega > 1$ rad/sec

$= 2 \times \frac{1}{2\pi} \int_1^\infty 3e^{-2\omega} = \frac{1}{\pi}\left[\frac{3}{2} e^{-2}\right]$

$\% = \dfrac{\frac{3}{2\pi} e^{-2}}{\frac{3}{2\pi}} = e^{-2}$ ← OK as final answer

7

$\% = \dfrac{\text{Power for } w > 1}{\text{Total power}} = \dfrac{2 \times \frac{1}{2\pi} \int_1^\infty e^{-2w}\,dw}{2 \times \frac{1}{2\pi} \int_0^\infty S_X(w)\,dw}$

EE508 Examination

Robert J. Marks II

Please write your name on the upper right hand corner.

This examination is closed book and closed notes. Calculators are allowed but will probably not be needed. Each student is allowed two $8\frac{1}{2}$ by 11 sheets of notes for the test.

All work will be done in this test booklet.

Each problem is worth 20 points.

Some hints:

- $7 \geq 2$
- $\exp(-\pi t^2)$ has a Fourier transform of $\exp\left(-\pi(\frac{\omega}{2\pi})^2\right)$.
- The Fourier transform of a unit step in time is

$$\pi\delta(\omega) + \frac{1}{j\omega}$$

- If the Fourier transform of $x(t)$ is $X(\omega)$, then the Fourier transform of $x(t-\tau)$ is $X(\omega)\exp(-j\omega\tau)$.
- The Fourier transform of a function $x(t)$ is defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- The inverse Fourier transform of $2\pi\delta(\omega)$ is one.
- The Fourier transform of the first derivative of $x(t)$ is $j\omega X(\omega)$.

1. **Fundamentals.** Circle the correst answer. Correct answers are +3, incorrect answers are -2 and no answer gives a zero score.

   (a)

   $$\overline{X^2} \geq \overline{X}^2$$

   TRUE FALSE

   (b) The power spectral density of real WSS stochastic processes is even.
   TRUE FALSE

   (c) Mean ergodic stochastic processes are also distribution ergodic.
   TRUE FALSE

   (d)

   $$F_X(x) \geq F_X(x-1)$$

   TRUE FALSE

   (e) $\overline{X}$ is always real.
   TRUE FALSE

   (f) All independent random variables are uncorrelated.
   TRUE FALSE

   (g) The characteristic function for the degenerate case of a deterministic "random variable" is always a complex exponential.
   TRUE FALSE

2. Let

$$Y(t) = \frac{d}{dt} X(t)$$

where $X(t)$ is a WSS process with a given $R_X(\tau)$ and $\overline{X(t)}$ = constant. What is $R_Y(\tau)$ and $\overline{Y(t)}$?

3. For the stochastic differential equation

$$Y(t) = \frac{d}{dt}X(t) + 2X(t),$$

find the autocorrelation, $R_Y(t_1, t_2)$, when $X(t)$ is a random walk with parameter α.

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2)$$

4. A Gaussian stochastic process, $X(t)$, and its (linear) Fourier transform are described as follows.

$$\overline{X(t)} = \mathrm{e}^{-\pi t^2}$$

$$R_X(t_1, t_2) = \mathrm{e}^{-\min(|t_1|, |t_2|)}$$

$$\chi(\omega) = \int_{-\infty}^{\infty} X(t)\mathrm{e}^{-j\omega t}dt$$

Find $\overline{\chi(\omega)}$.

5. The autocorrelation of a stochastic process is

$$R(t+\tau,\tau) = \cos\left(\frac{2\pi(t+\tau)}{T}\right) \times \mathrm{e}^{-|\tau|}$$

(a) Is the stochastic process cyclo-stationary? Show why the answer you give is correct.

(b) If the stochastic process is cyclostationary, evaluate the autocorrelation of a process formed by randomizing the origin on the interval $(0, T]$

6. A *whitening filter* is a linear time invariant filter that transforms an input, $X(t)$, into a white stochastic process with autocorrelation, $R_Y(\tau) = q\delta(\tau)$. Specify the frequency response, $H(\omega)$, of a whitening filter when the autocorrelation of the input is

$$R_X(\tau) = \mathrm{e}^{-\pi\tau^2}.$$

7. A digital filter has a transfer function equal to

$$H(z) = \frac{1}{z - \frac{1}{2}}$$

The input to the filter is discrete white noise with autocorrelation

$$R_X[n] = q\ \delta[n]$$

Let the filter output be $Y[n]$. Find the power spectral density, $S_Y(e^{j\omega})$, of the output. Your answer should be real and non-negative.

1. scratch paper

2. scratch paper

3. scratch paper

EE505 Final Examination

Robert J. Marks II

December 13, 1996; 2:20 to 4:30 PM
Special Friday the Thirteenth Final Exam!!

The examination is closed book and closed notes. Calculators are allowed. Each student is allowed three sheets of notes. All problems are weighted equally.

All work will be done in a test booklet. No scratch paper is needed.

1. The events A and B are independent. Are the events $\bar{A}$ and $\bar{B}$ always independent? If not, provide a counter example.

2. A die is rolled. The stochastic process, $X(t)$, is set to $5\ \mathrm{e}^{-t}$ when the die shows six dots. Otherwise, the stochastic process is set equal to $-\mathrm{e}^{-t}$.

   (a) Evaluate the expected value of $X(t)$.

   (b) Evaluate the autocorrelation, $R_X(t_1, t_2)$.

3. Let X be a Poisson random variable with unknown parameter a. Assume a is sufficiently large to apply the central limit theorem. We perform an experiment the outcome of which is 14. Estimate a 98% confidence interval for the parameter a. Assume $2\ \mathrm{erf}(2) = 0.98$.

4. Consider, on a plane, a small area a within a larger area A. We place n points at random within the large area, A.

   (a) Let k be the number of points in the small area a. Evaluate the density function for k assuming the area a is much smaller than the A using the Poisson approximation.

   (b) Let both the area A and the number of points, n, go to infinity such that the ratio of n to the area is a constant, λ. The result is a two dimensional Poisson process. For a given λ, what is the radius of a circle such that the probability of containing no points is one half? Express your answer in terms of λ.

5. The stationary stochastic process $X(t)$ is zero mean with a uniform distribution. Its autocorrelation is

$$R_X(\tau) = \frac{1}{12} e^{-a|\tau|}.$$

(a) What percentage of the time is $|X(t)| \leq 0.3$?

(b) Evaluate the expected value of

$$12 \left[X(t) + X\left(t + \frac{1}{a}\right) \right]^2$$

4

6. Let the random variables $\{X_n | 0 \le n \le N\}$ are uniformly distributed on the interval (0,1) and are *i.i.d.* Define the random variable

$$Y = \prod_{n=1}^{N} X_n.$$

Estimate the probability $Y \le \exp(-3)$. Assume $N = 9$ is sufficiently large to apply the central limit theorem to $\ln(Y)$.[1]

[1] $\int_0^1 \ln(z)dz = -1$; $\int_0^1 \ln^2(z)dz = 2$

$\mathrm{erf}(2) = 0.48$

1. Scratch Paper

2. Scratch Paper

3. Scratch Paper

4. Scratch Paper

EE508 Take-Home Final Examination

Instructions.

- Attach this test booklet to your exam.
- You may use any nonhuman resource except for Professor Marks. The EE Department's honor code is in force.
- For take home exams, neatness counts.
- Each problem is worth 20 points.
- The exam must be turned in before March 20 at 5PM.

You may find the following Fourier transform pairs useful

$$e^{-|\tau|} \Leftrightarrow \frac{2}{1+\omega^2}.$$

$$\text{If } x(t) \Leftrightarrow X(\omega), \text{ then } x(at) \Leftrightarrow \frac{1}{|a|} X(a\omega)$$

1. The autocorrelation of a zero mean stochastic process is

$$R_X(\tau) = e^{-\alpha|\tau|}$$

Design a linear time invariant filter using adders, differentiators and amplifiers/attenuators so that, when $X(t)$ is fed into the filter, the output, $Y(t)$, is white.

2. A European call option is priced at a $r = 0$ safe investment rate. The strike price is set to the current securities price. In other words, $X = S$. If the call option has a duration of $T =$ two months, what is the implied volatility, σ, when the Black-Scholes formula places the fair price of the option at 10% of the strike price?

3. Differentiating the sampling theorem expansion for a deterministic signal, $x(t)$, with bandwidth B, gives

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} x(nT) \frac{d}{dt} \frac{\sin \sigma(t-nT)}{\sigma(t-nT)}$$

where $\sigma = 2\pi B$ and $T = \pi/\sigma$. Let $X(t)$ be a σ-bandlimited stochastic process. Does

$$\frac{dX(t)}{dt} = \sum_{n=-\infty}^{\infty} X(nT) \frac{d}{dt} \frac{\sin \sigma(t-nT)}{\sigma(t-nT)}$$

in the mean square sense?

4. Define $X(t) = \sum_{n=-N}^{N} A_n \, e^{j2\pi nt/T}$ where T is a given period and the $2N+1$ random variables, A_n, are independent and identically distributed random variables with real mean a and variance σ^2.

   - Compute the mean of $X(t)$. Express your (real) answer as the ratio of two sin functions.
   - Evaluate the autocorrelation of $X(t)$. Is $X(t)$ cyclostationary?
   - Let $Y(t) = X(t - \Theta)$ where Θ is a uniform random variable on the interval $(0, T)$. Is $Y(t)$ ergodic in the mean?

5. Let $X[n]$ denote a real discrete stationary Gaussian stochastic process with mean η_X and variance $\mathrm{var}(X)$. Let K denote a binomial random variable corresponding to N trials with a probability of success p. Let K and X be independent. Consider the sum

$$S = \sum_{k=0}^{K} X^2[n]$$

Evaluate the expected value of the random variable S.

EE508 Midterm Examination #2

Winter 1997

Instructions.

- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed two notebook sized sheet of notes & a calculator.
- Each problem is worth 20 points.

1 Problem

A common model for the autocovariance of certain stochastic processes is

$$C_X(\tau) = \mathrm{var}(X)\, \mathrm{e}^{-\lambda|\tau|} \cos(\omega_0 \tau)$$

Does the integral

$$\frac{1}{2T} \int_{-T}^{T} X(t) dt$$

converge to the (non-zero) mean of $X(t)$? In other words, is $X(t)$ ergodic in the mean?

2 Problem

A securities price, S, follows the familiar Ito process

$$dS = \mu S dt + \sigma S dz.$$

Describe the Ito process for S^2.

3 Problem

Consider the first order difference equation

$$Y[n-1] + 2Y[n] = 4X[n]$$

What is the power spectral, $S_Y(e^{j\omega})$, when the input is discrete white noise with power spectral density of $S_X(e^{j\omega}) = q$?

4 Problem

A stochastic process, $X(t)$, has an autocorrelation of

$$R_X(t_1, t_2) = e^{-|t_1|} + e^{-|t_2|}$$

Let $Y(\omega)$ be the Fourier transform of $X(t)$.

$$Y(\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

Evaluate the autocorrelation, $R_Y(\omega_1, \omega_2)$ of the stochastic process $Y(\omega)$.

5 Problem

Let $X(t)$ denote a Wiener process with parameter α. Define the stochastic process

$$Z(t) = X(t) - X(t - \tau)$$

where τ is a given positive number. Determine the first order probability density function of $Z(t)$ for all positive values of t.

Scratch Paper #1

Scratch Paper #2

Scratch Paper #3

Scratch Paper #4

EE508 Quiz #1

____
(name)

_______________ /100
(score)

All four problems are worth 25 points each. For problems 2-4, show your work. Closed notes & books. You are allowed one page of notes. The test is designed for 1 hr, but you may take the entire 2 hrs to do it.

1. Match the stochastic process sample to the best description. Use an answer only once. No penalty for guessing.

   a. $E[X(t)] \neq$ constant ______ ← write your answers here

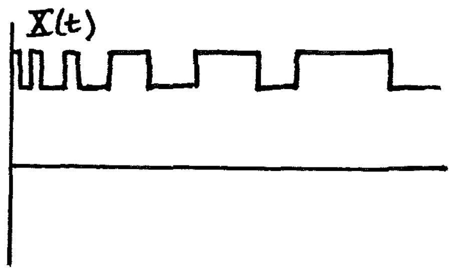
   b. $R_X(t_1, t_2) \neq R_X(t_1 - t_2)$ ______

   c. var $X(t) \neq$ constant ______

   d. cyclostationary ______

   e. non-ergodic .... ______

   f. stationary .... ______

A.


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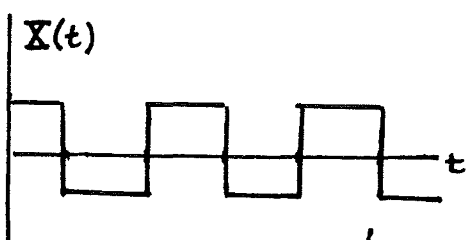

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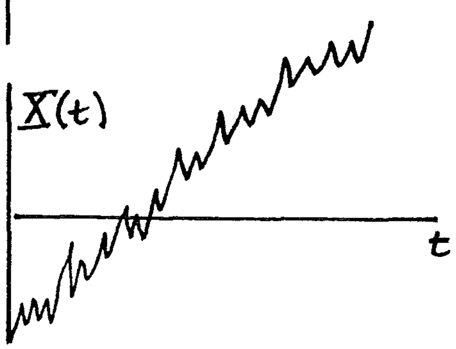

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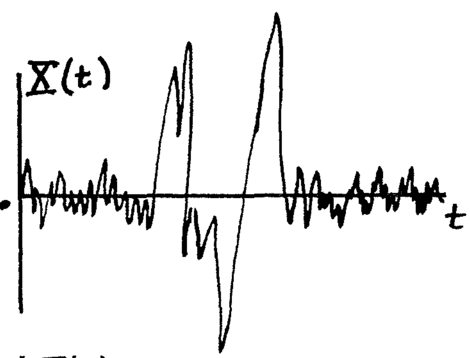

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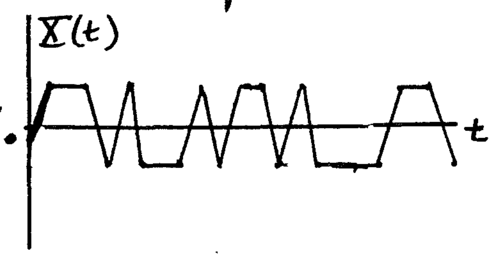
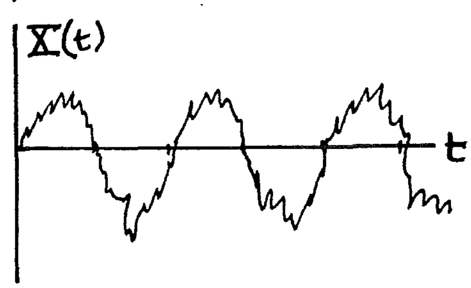


F.


$X(t)$ is a zero mean gaussian (normal) process with autocorrelation:

$$R_X(\tau) = \left(\frac{\sin \pi\tau}{\pi\tau}\right)^2$$

(a) Let $Y = X(\frac{3}{2}) + X(\frac{9}{2}) + X(\frac{21}{2})$. Find the probability density function for Y.

(b) Find the probability that

$$X(\tfrac{1}{3})\, X(\tfrac{10}{3}) > 0$$

Let $\{a_n \mid -\infty < n < \infty\}$ be uncorrelated random variables with zero mean and variance $\sigma_n^2 = \sigma^2$ for all n. Consider

$$X(t) = \sum_{n=-\infty}^{\infty} a_n \, \delta(t - nT)$$

where T is a given parameter.

(a) Is $X(t)$ wide sense cyclostationary? (Show your work)

(b) Let Θ denote a random variable uniform on $(0, T)$ and independent of the a_n's. Is $Y(t) = X(t - \Theta)$ ergodic in the mean?

The Fourier Transform is a linear operation. Consider

$$\mathbf{Y}(\omega) = \int_{-\infty}^{\infty} \mathbf{X}(t)\, e^{-j\omega t}\, dt$$

(a) Find the autocorrelation for $\mathbf{Y}(\omega)$ $[R_{\mathbf{Y}}(\omega_1, \omega_2)]$ when $\mathbf{X}(t)$ is (zero mean) white noise:

$$R_{\mathbf{X}}(t_1, t_2) = q(t_1)\, \delta(t_1 - t_2)$$

(b) What type of process is $\mathbf{Y}(\omega)$?
(Let $Q(\omega) = \int_{-\infty}^{\infty} q(t)\, e^{-j\omega t}\, dt$ in (a))

EE508
Final Exam

Name ____________________
Score ____________ /200

Rules:

1. No human help (other than Prof. Marks) is allowed.
2. Due 9 A.M., Mon March 17th. Please have your test placed in Prof. Marks' mailbox. Do <u>not</u> slip under his office door.
3. In final form, your exam should be stapled with this page as a cover sheet.
4. Since this is a take home test, neatness counts.
5. Sign and date the statement below:

"All of the references I have used for this test (human or otherwise) are listed on the back of this page"

X ____________________
SIGN

____
DATE

Each problem is worth 100 pts.

"... of making many books there is no end; and much study is a weariness of the flesh."

King Solomon in Eccl. 12:12

LINE CODING

1. Let b_n denote iid samples from the density:

$$f_b(x) = \tfrac{1}{2}\,\delta(x+1) + \tfrac{1}{2}\,\delta(x-1)$$

A sequence of b_n's could be considered a stream of (bipolar) bits.

(a) Show that $X(t) = \sum_n b_n\,\delta(t-nT)$ is cyclostationary. Then, compute the power spectral density of

$$Y(t) = X(t-\Theta)$$

where Θ is uniform on $(0,T)$

(b) Find the power spectral density of*

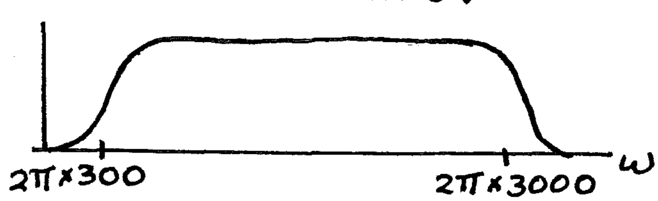
$$Z(t) = \sum_n b_n\, \mathrm{rect}\left(\frac{t-\Theta-nT}{T}\right)$$

Note that $Z(t)$ is the output of a filter with impulse response

$$h(t) = \mathrm{rect}\,(t/T)$$

when $Y(t)$ is input. $Z(t)$ is a "polar coding" of the bit stream. Sketch $S_Z(\omega)$ vs. ω.

(c) The bit stream in (b) could, for example, be sent over a phone line. Unfortunately, phone lines typically have a frequency response like this:



The power spectral density in (b) has significant low frequency components which can't be sent over the line. A line coding technique (commonly used) that gets around this problem is bipolar line coding. (cont) →

* rect ξ = 1 for $|\xi| < \tfrac{1}{2}$ and is otherwise zero.

1 (c) (cont). In bipolar coding, both a +1 and a -1 represent the binary 1 and 0 represents 0. If a binary 1 is represented as a -1, the next binary 1 is represented as a +1 (and visa versa). Here's an example:

bits:	0	1	1	0	0	1	1	1	0	1	
binary code:	-1	1	1	-1	-1	1	1	1	-1	1	: b_n
bipolar code:	0	1	-1	0	0	1	-1	1	0	-1	: c_n

Equivalently, every other 1 in the bit stream is negated.

The bipolar code can be viewed as a Markoff process if we treat the bipolar zeros in two ways: A 0^+ is a zero that will convert to a +1 the next time a 1 bit occurs. A 0^- will convert to a -1. Thus, only a -1 or a 0^- can follow a 0^-.

Using the four states 0^+, 0^-, $\underline{1}$ and -1, draw the state diagram and Π matrix for this process (as in Fig 12·14 of your text).

(d) For what initial state probabilities is this homogeneous Markoff process stationary?

(e) Let $\{C_n\}$ denote the bipolar code of corresponding binary code, $\{b_n\}$. Show that C_n is independent of C_{n+m} if $|m| > 1$. [Hint: Look at Π^m for $m > 1$]. Assume stationarity.

(f) Let $X(t) = \sum_{n=-\infty}^{\infty} C_n \delta(t - nT)$, and $Y(t) = X(t - \Theta)$.

Find the power spectral density of $Y(t)$.

(g) Let $Z(t) = \sum_{n=-\infty}^{\infty} C_n \, rect\left(\frac{t - \Theta - nT}{T}\right)$. Find the power spectral density of $Z(t)$ and note that this line coding technique can fit our phone line better than the binary case.

JITTER

2. (a) Let $\{X_n \mid -\infty < n < \infty\}$ denote iid samples from a density $f_X(x)$. Let $t_n = nT + X_n$ where T is a specified sampling interval. Define

$$Y(t) = \sum_n \delta(t - t_n)$$

Show that $Y(t)$ is wide-sense cyclostationary.

(b) Let Θ be a uniform random variable on $(0, T)$ that is independent of the X_n's. Define

$$Z(t) = Y(t - \Theta)$$

Derive the mean of $Z(t)$ and show that:

$$C_Z(\tau) = \frac{1}{T}\delta(\tau) + \frac{1}{T}\sum_{n \neq 0} r_X(\tau - nT) - \frac{1}{T^2}$$

where

$$r_X(\tau) = \int_{-\infty}^{\infty} f_X(t)\, f_X(t + \tau)\, d\tau$$

Hints: • For any function $f(t)$:

$$\sum_n \int_0^T f(t - nT)\, dt = \int_{-\infty}^{\infty} f(t)\, dt$$

- Also, I found the Poisson sum formula useful.
- If $H(u) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ut} dt$, $h(t)$ is real, and $r_h(t) = \int_{-\infty}^{\infty} h(\tau) h(t + \tau) d\tau$ then

$$\int_{-\infty}^{\infty} r_h(t) e^{-j2\pi ut} dt = |H(u)|^2$$

(c) Define

$$\tilde{G}(u) = T \sum_n g(\hat{t}_n) e^{-j2\pi u \hat{t}_n} \; ; \; \hat{t}_n = t_n + \Theta$$

Show that $\tilde{G}(u)$ is an unbiased estimate of

$$G(u) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ut} dt$$

2. (cont)

(d) Let $g(t)$ be real with energy:

$$E = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

Furthermore, assume $g(t)$ is bandlimited with maximum frequency W:

$$G(u) = 0 \quad ; \quad |u| > W$$

Show that, if $T \leq 1/2W$, then ⊛

$$\text{var}\,\tilde{G}(u) = TE - T\,|G(u)|^2 * |\Phi_X(u)|^2 ; |u| < W$$

where $*$ denotes convolution.

(e) Interpret your results in (d) when there is no jitter. That is, each X_n is equal to a deterministic zero.

(f) Compare this estimate to the one where $g(t)$ is sampled at Poisson points with density $\lambda = 1/T$. [pp. 335-338 of Papoulis] Specifically, which gives the best estimate of $G(u)$?

⊛ $\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x) e^{-j2\pi ux} dx$ = characteristic function

EE508
Final Exam

Name ____________________
Score ____________ /200

Rules:

1. No human help (other than Prof. Marks) is allowed.
2. Due 9 A.M., Mon March 17th. Please have your test placed in Prof. Marks' mailbox. Do not slip under his office door.
3. In final form, your exam should be stapled with this page as a cover sheet.
4. Since this is a take home test, neatness counts.
5. Sign and date the statement below:

"All of the references I have used for this test (human or otherwise) are listed on the back of this page"

X ____________________
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____
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Each problem is worth 100 pts.

"... of making many books there is no end; and much study is a weariness of the flesh."

King Solomon in Eccl. 12:12

2. (cont)

(d) Let $g(t)$ be real with energy:

$$E = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

Furthermore, assume $g(t)$ is bandlimited with maximum frequency W:

$$G(u) = 0 \;;\; |u| > W$$

Show that, if $T \leq 1/2W$, then⊛

$$\operatorname{var} \tilde{G}(u) = TE - T\,|G(u)|^2 * |\Phi_X(u)|^2 \;;\; |u| < W$$

where $*$ denotes convolution.

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⊛ $\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x)\, e^{-j2\pi ux} dx$ = characteristic function

JITTER

2. (a) Let $\{X_n \mid -\infty < n < \infty\}$ denote iid samples from a density $f_X(x)$. Let $t_n = nT + X_n$ where T is a specified sampling interval. Define

$$Y(t) = \sum_n \delta(t - t_n)$$

Show that $Y(t)$ is wide-sense cyclostationary.

(b) Let Θ be a uniform random variable on $(0,T)$ that is independent of the X_n's. Define

$$Z(t) = Y(t - \Theta)$$

Derive the mean of $Z(t)$ and show that:

$$C_Z(\tau) = \frac{1}{T}\delta(\tau) + \frac{1}{T}\sum_{n \neq 0} r_X(\tau - nT) - \frac{1}{T^2}$$

where

$$r_X(\tau) = \int_{-\infty}^{\infty} f_X(t)\, f_X(t+\tau)\, d\tau$$

Hints: • For any function $\varphi(t)$:

$$\sum_n \int_0^T \varphi(t - nT)\, dt = \int_{-\infty}^{\infty} \varphi(t)\, dt$$

- Also, I found the Poisson sum formula useful.
- If $H(u) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi u t} dt$, $h(t)$ is real, and $r_h(t) = \int_{-\infty}^{\infty} h(\tau) h(t+\tau) d\tau$ then

$$\int_{-\infty}^{\infty} r_h(t) e^{-j2\pi u t} dt = |H(u)|^2$$

(c) Define

$$\tilde{G}(u) = T \sum_n g(\hat{t}_n) e^{-j2\pi u \hat{t}_n}; \quad \hat{t}_n = t_n + \Theta$$

Show that $\tilde{G}(u)$ is an unbiased estimate of

$$G(u) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi u t} dt$$

1 (c) (cont). In bipolar coding, both a +1 and a −1 represent the binary 1 and 0 represents 0. If a binary 1 is represented as a −1, the next binary 1 is represented as a +1 (and visa versa). Here's an example:

bits:	0	1	1	0	0	1	1	1	0	1	
binary code:	−1	1	1	−1	−1	1	1	1	−1	1	: b_n
bipolar code:	0	1	−1	0	0	1	−1	1	0	−1	: c_n

Equivalently, every other 1 in the bit stream is negated.

The bipolar code can be viewed as a Markoff process if we treat the bipolar zeros in two ways: A 0^+ is a zero that will convert to a +1 the next time a 1 bit occurs. A 0^- will convert to a −1. Thus, only a −1 or a 0^- can follow a 0^-.

Using the four states 0^+, 0^-, 1 and −1, draw the state diagram and Π matrix for this process (as in Fig 12·14 of your text).

(d) For what initial state probabilities is this homogeneous Markoff process stationary?

(e) Let $\{C_n\}$ denote the bipolar code of corresponding binary code, $\{b_n\}$. Show that C_n is independent of C_{n+m} if $|m|>1$. [Hint: Look at Π^m for $m>1$]. Assume stationarity.

(f) Let $X(t) = \sum_{n=-\infty}^{\infty} c_n \delta(t-nT)$, and $Y(t) = X(t-\Theta)$.

Find the power spectral density of $Y(t)$.

(g) Let $Z(t) = \sum_{n=-\infty}^{\infty} c_n \, rect\left(\frac{t-\Theta-nT}{T}\right)$. Find the power spectral density of $Z(t)$ and note that this line coding technique can fit our phone line better than the binary case.

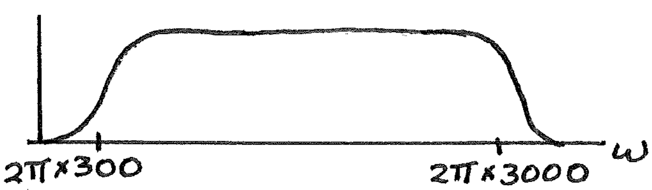
LINE CODING

1. Let b_n denote iid samples from the density:
$$f_b(x) = \tfrac{1}{2}\,\delta(x+1) + \tfrac{1}{2}\,\delta(x-1)$$
A sequence of b_n's could be considered a stream of (bipolar) bits.

(a) Show that $X(t) = \sum_n b_n\,\delta(t-nT)$ is cyclostationary. Then, compute the power spectral density of
$$Y(t) = X(t-\Theta)$$
where Θ is uniform on $(0,T)$

(b) Find the power spectral density of*
$$Z(t) = \sum_n b_n\,\mathrm{rect}\left(\frac{t-\Theta-nT}{T}\right)$$
Note that $Z(t)$ is the output of a filter with impulse response
$$h(t) = \mathrm{rect}(t/T)$$
when $Y(t)$ is input. $Z(t)$ is a "polar coding" of the bit stream. Sketch $S_Z(\omega)$ vs. ω.

(c) The bit stream in (b) could, for example, be sent over a phone line. Unfortunately, phone lines typically have a frequency response like this:



The power spectral density in (b) has significant low frequency components which can't be sent over the line. A line coding technique (commonly used) that gets around this problem is bipolar line coding. (cont) →

* $\mathrm{rect}\,\xi = 1$ for $|\xi| < \frac{1}{2}$ and is otherwise zero.

Solutions

1.(a) $X(t) = \sum_{n=-\infty}^{\infty} b_n \, \delta(t - nT)$; $E\,X(t) = 0$

$R_X(t;\tau) = E\,X(t)\,X(\tau)$

$= \sum_n \sum_m \delta(t-nT)\,\delta(\tau - mT)\, E\, b_n b_m$

But, since b_n & b_m are ind. for $n \neq m$:

$E\, b_n b_m = E\, b_n\, E\, b_m = 0$; $n \neq m$

Thus: $E\, b_n b_m = \delta[n-m]$ and:

$R_X(t;\tau) = \sum_n \delta(t - nT)\,\delta(\tau - nT)$

$R_X(t+\tau, t) = \sum_n \delta(t+\tau - nT)\,\delta(t - nT)$

$\Rightarrow$ cyclostationary

$Y(t) = X(t - \Theta)$; Θ uniform on $(0, T)$

$\Rightarrow R_Y(\tau) = \frac{1}{T}\int_0^T R_X(t+\tau, t)\,dt$

$= \frac{1}{T} \sum_n \int_{-T/2}^{T/2} \delta(t+\tau-nT)\,\delta(t-nT)\,dt$

For $\delta(t-nT)$ to be in $(-T/2, T/2)$, $n=0$:

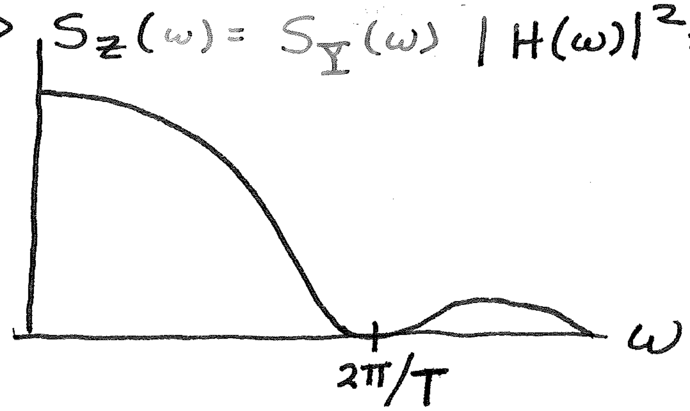
$R_Y(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t+\tau)\,\delta(t)\,dt$

$= \frac{1}{T}\,\delta(\tau) \Rightarrow S_Y(\omega) = \frac{1}{T}$

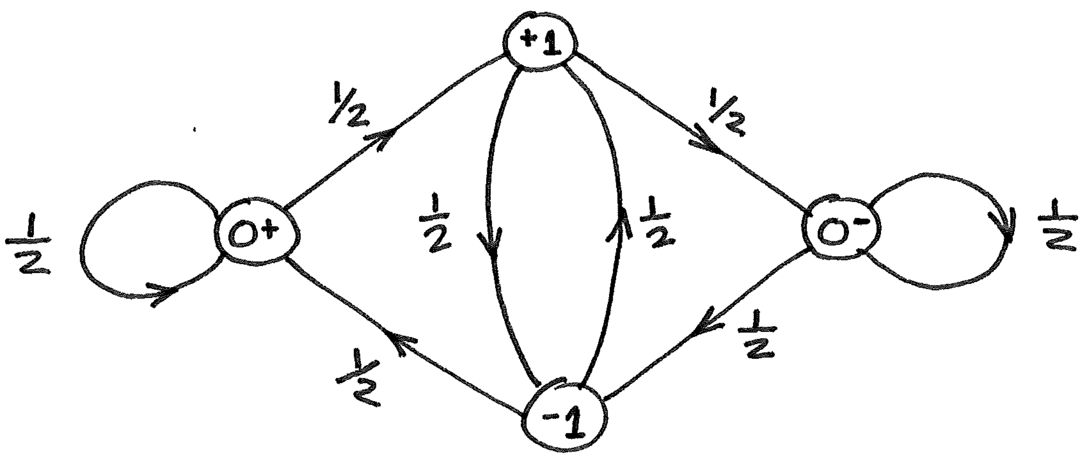
(b) $Y(t) \longrightarrow \boxed{h(t) = rect\,(t/T)} \longrightarrow Z(t)$

$H(\omega) = \dfrac{2 \sin \frac{\omega T}{2}}{\omega}$

$\Rightarrow S_Z(\omega) = S_Y(\omega)\,|H(\omega)|^2 = \frac{4}{T}\left(\dfrac{\sin \frac{\omega T}{2}}{\omega}\right)^2$



(c) $C_n = \begin{cases} 0^+ & \text{if } C_{n-1} = 0^+ \text{ or } -1 \text{ and } b_n = 0 \\ 0^- & \text{" } C_{n-1} = 0^- \text{ or } +1 \text{ " } b_n = 0 \\ +1 & \text{" } C_{n-1} = 0^+ \text{ or } -1 \text{ " } b_n = 1 \\ -1 & \text{" } C_{n-1} = 0^- \text{ or } +1 \text{ " } b_n = 1 \end{cases}$



$$\begin{array}{c} \\ \Pi = \end{array} \begin{array}{c} \begin{array}{cccc} 0^+ & 0^- & 1 & -1 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{GIVEN} \\ 0^+ \\ 0^- \\ 1 \\ -1 \end{array}$$

(e) Π^n, for $n > 1$, has all elements equal to $1/4$. Thus

$$P[C_{n+m} \mid C_n] = P[C_{n+m}] \text{ for } m > 1.$$

(d) Since $\left[\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\right]^T = \Pi\left[\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\right]^T$,

the process is stationary for

$$P[0^-] = P[0^+] = P[1] = P[-1] = 1/4$$

(f) $X(t) = \sum_n c_n \, \delta(t - nT)$

$E X(t) = 0$ since $E c_n = 0$ for the stationary case.

$$R_X(t;\tau) = E X(t) X(\tau)$$

$$= \sum_n \sum_m \delta(t - nT)\, \delta(\tau - mT)\, E c_n c_m$$

If $n = m$, $E c_n^2 = [(-1)^2 + (1)^2 + (0^+)^2 + (0^-)^2]\, 1/4$

$$= 1/2$$

If $m = n+1$, $c_n c_{n+1}$ is either 0 or -1 (Can't be 1 since two ± 1's can't be back to back)

Could be: $(0^+ 0^+)$, $(0^+ 1)$
$(0^- 0^-)$, $(0^-, -1)$
$(1\ 0^-)$, $(1, -1)$
$(-1, 0^+)$, $(-1, 1)$

Each pair with equal probability. Thus:

$$E c_n c_{n+1} = -1/4$$

Similarly, $E c_n c_{n-1} = -1/4$

For all other cases, $E c_n c_m = E c_n E c_m = 0$

And:

$$R_X(t;\tau) = \sum_n \sum_m \delta(t - nT)\, \delta(\tau - mT) \left[\tfrac{1}{2} \delta[n-m] - \tfrac{1}{4} \delta[n-m-1] - \tfrac{1}{4} \delta[n-m+1]\right]$$

$$= \sum_n \tfrac{1}{2} \delta(t - nT)\, \delta(\tau - nT)$$
$$- \tfrac{1}{4} \delta(t - nT)\, \delta(\tau - (n+1)T)$$
$$- \tfrac{1}{4} \delta(t - nT)\, \delta(\tau - (n-1)T)$$

Thus:

$$R_X(t+\tau;t) = \sum_n \delta(t+\tau-nT)\left[\frac{1}{2}\delta(t-nT) - \frac{1}{4}\delta(t-(n+1)T) - \frac{1}{4}\delta(t-(n-1)T)\right]$$

$\therefore$ X is cyclostationary since this is a periodic function of t.

$$Y(t) = X(t-\Theta)$$

$$R_Y(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \sum_n \delta(t+\tau-nT)\left[\frac{1}{2}\delta(t-nT) - \frac{1}{4}\delta(t-(n+1)T) - \frac{1}{4}\delta(t-(n-1)T)\right]dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[\frac{1}{2}\delta(t-\tau)\,\frac{1}{2}\delta(t) - \frac{1}{4}\delta(t+\tau+T)\,\delta(t) - \frac{1}{4}\delta(t+\tau-T)\,\delta(t)\right]dt$$

$$= \frac{1}{T}\left[\frac{1}{2}\delta(\tau) - \frac{1}{4}\delta(\tau+T) - \frac{1}{4}\delta(\tau-T)\right]$$

$$S_Y(\omega) = \frac{1}{T}\left[\frac{1}{2} - \frac{1}{4}e^{j\omega T} - \frac{1}{4}e^{-j\omega T}\right]$$

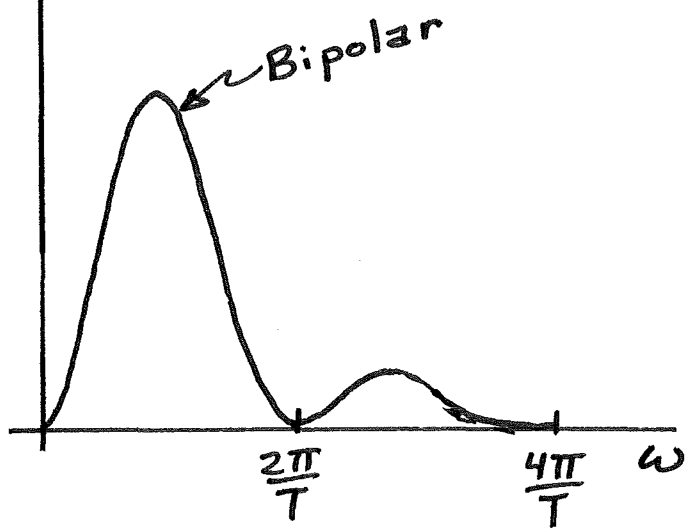
$$= \frac{1}{T}\left[\frac{1}{2} - \frac{1}{2}\cos\omega T\right]$$

$$= \frac{1}{T}\sin^2\frac{\omega T}{2}$$

(g) Using the same filter as in (b):

$$S_Z(\omega) = \frac{1}{T}\sin^2\frac{\omega T}{2} \times \left(\frac{2\sin\frac{\omega T}{2}}{\omega}\right)^2$$

$$= \frac{4}{T\omega^2}\sin^4\frac{\omega T}{2}$$



2 (a) $\mathbf{Y}(t) = \sum_n \delta(t - t_n)$

$t_n = nT + \mathbf{X}_n \ , \ \mathbf{X}_n \sim f_{\mathbf{X}}(x) \quad \text{(iid)}$

$$E\,\mathbf{Y}(t) = \eta_y(t) = \sum_n E\,\delta(t - nT - \mathbf{X}_n)$$
$$= \sum_n \int_{-\infty}^{\infty} f_{\mathbf{X}}(x)\,\delta[x - (t - nT)]$$
$$= \sum_n f_{\mathbf{X}}(t - nT) \qquad (1)$$

$$R_{\mathbf{Y}}(t;\tau) = E\,\mathbf{Y}(t)\,\mathbf{Y}(\tau)$$
$$= \sum_n \sum_m E\ \delta(t - t_n)\,\delta(\tau - t_m)$$
$$= \sum_n \Big[E\,\delta(t - t_n)\,\delta(\tau - t_n) + \sum_{m \neq n} E\,\delta(t - t_n)\,\delta(\tau - t_m) \Big]$$
$$= \sum_n \Big[\int_{-\infty}^{\infty} f_{\mathbf{X}}(x)\,\delta\{x - (t - nT)\}\,\delta(x - (\tau - nT))\,dx + \sum_{m \neq n} \int_x \int_y f_{\mathbf{X}}(x)\,f_{\mathbf{X}}(y) \times \delta\{x - (t - nT)\}\,\delta\{y - (\tau - mT)\}\,dx\,dy$$
$$= \sum_n \Big[f_{\mathbf{X}}(t - nT)\,\delta(t - \tau) + \sum_{m \neq n} f_{\mathbf{X}}(t - nT)\,f_{\mathbf{X}}(\tau - nT) \Big]$$

$$\Rightarrow R_{\mathbf{Y}}(t + \tau; t) = \sum_n f_{\mathbf{X}}(t - nT)\,\delta(\tau) + \sum_n \sum_{m \neq n} f_{\mathbf{X}}(t + \tau - nT)\,f_{\mathbf{X}}(t - mT) \qquad (2)$$

To be cyclostationary in the wide sense, we need to satisfy two conditions:

① $n_Y(t+pT) = n_Y(t)$; $p \in$ Integer

② $R_Y(t+\tau+pT, t+pT) = R_Y(t+\tau; t)$

These conditions are satisfied by Eqs. (1) and (2) respectively

(b) $Z(t) = Y(t-\Theta)$; Θ uniform on $(0,T)$

$$\Rightarrow n_Z = \frac{1}{T}\int_0^T n_Y(t)\,dt$$

$$= \frac{1}{T}\sum_n \int_0^T f_X(t-nT)\,dt$$

For any function, $f(t)$:

$$\sum_n \int_0^T f(t-nT)\,dt = \int_{-\infty}^{\infty} f(t)\,dt \qquad (3)$$

Thus:

$$n_Z = \frac{1}{T}\int_{-\infty}^{\infty} f_X(t)\,dt = 1/T \qquad (4)$$

$$\Rightarrow R_Z(\tau) = \frac{1}{T}\int_0^T R_Y(t+\tau, t)\,dt$$

$$= \frac{1}{T}\delta(\tau)\sum_n \int_0^T f_X(t-nT)\,dt$$

$$+ \frac{1}{T}\sum_n \sum_{m\neq n} \int_0^T f_X(t+\tau-nT)\, f_X(t-mT)\,dt$$

$$= \frac{1}{T}\delta(\tau)$$

$$+ \frac{1}{T}\sum_n \sum_m \int_0^T f_X(t+\tau-nT)\, f_X(t-mT)\,dt$$

$$- \frac{1}{T}\sum_n \int_0^T f_X(t+\tau-nT)\, f_X(t-nT)\,dt$$

where we have used (3). Continuing:

$$R_z(\tau) = \frac{1}{T}\delta(\tau) + \frac{1}{T}\int_0^T \sum_n f_X(t+\tau-nT) \sum_m f_X(t-mT)\,dt$$
$$- \frac{1}{T}\int_{-\infty}^{\infty} f_X(t+\tau)\, f_X(t)\,dt \qquad (5)$$

where we have again used (3) with
$$f(t) = f_X(t+\tau)\, f_X(t)$$
Define the characteristic function:
$$\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x)\, e^{-j2\pi u x}\,dx \qquad (6)$$
Then, from the Poisson sum formula:
$$T \sum_n f_X(t-nT) = \sum_n \Phi_X\left(\frac{n}{T}\right) e^{j2\pi nt/T} \qquad (7)$$
Using this in (5):
$$R_z(\tau) = \frac{1}{T}\delta(\tau) + \frac{1}{T^3}\int_0^T \sum_n \Phi_X\left(\frac{n}{T}\right) e^{j2\pi n(t-\tau)/T}$$
$$\times \sum_m \Phi_X\left(\frac{m}{T}\right) e^{j2\pi mt/T}$$
$$- \frac{1}{T} r_X(\tau)$$

where
$$r_X(\tau) = \int_{-\infty}^{\infty} f_X(t+\tau)\, f_X(t)\,dt$$
$$\leftrightarrow |\Phi_X(u)|^2 \qquad (8)$$

Continuing:

$$R_Z(\tau) = \frac{1}{T}\delta(\tau) - \frac{1}{T} r_X(\tau)$$
$$+ \frac{1}{T^3} \sum_n \sum_m \Phi_X\left(\frac{n}{T}\right) \Phi_X\left(\frac{m}{T}\right) e^{-j2\pi n\tau/T}$$
$$\times \int_0^T e^{j2\pi(n+m)t/T} dt$$

But $\int_0^T e^{j2\pi pt/T} dt = T\,\delta[p]$ and

$$R_Z(\tau) = \frac{1}{T}\delta(\tau) - \frac{1}{T} r_X(\tau)$$
$$+ \frac{1}{T^2} \sum_n \left| \Phi_X\left(\frac{n}{T}\right)\right|^2 e^{-j2\pi n\tau/T}$$

Using (8) and the Poisson sum formula gives:

$$R_Z(\tau) = \frac{1}{T}\delta(\tau) - \frac{1}{T} r_X(\tau) + \frac{1}{T}\sum_n r_X(\tau - nT)$$
$$= \frac{1}{T}\delta(\tau) + \frac{1}{T}\sum_{n\neq 0} r_X(\tau - nT) \qquad (9)$$

Then:

$$C_Z(\tau) = R_Z(\tau) - \eta_Z^2$$
$$= \frac{1}{T}\delta(\tau) + \frac{1}{T}\sum_{n\neq 0} r_X(\tau - nT) - \frac{1}{T^2} \qquad (10)$$

(c) $\tilde{G}(u) = T \sum_n g(\hat{t}_n) e^{-j2\pi u \hat{t}_n}$; $\hat{t}_n = t_n + \Theta$

or $\tilde{G}(u) = T \int_{-\infty}^{\infty} g(t) e^{-j2\pi ut} Z(t)\, dt$; g real

Thus:

$$E\, \tilde{G}(u) = T \int_{-\infty}^{\infty} g(t) e^{-j2\pi ut} E\, Z(t)\, dt$$
$$= G(u)$$

where we have used (4) and

$$G(u) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ut} dt$$

Thus, $\tilde{G}$ is an unbiased estimate of G.

(d) $\text{var}\, \tilde{G}(u) = E \left| \tilde{G}(u) - G(u) \right|^2$

$$= E \left| T \int_{-\infty}^{\infty} g(t) e^{-j2\pi ut} \left\{ Z(t) - \tfrac{1}{T} \right\} dt \right|^2$$

$$= T^2 \int_t \int_\tau g(t) g(\tau) e^{-j2\pi u(t-\tau)} \times E\left\{ Z(t) - \tfrac{1}{T} \right\} \left\{ Z(\tau - \tfrac{1}{T}) \right\} dt\, d\tau$$

but

$$E\left\{ Z(t) - \tfrac{1}{T} \right\} \left\{ Z(\tau) - \tfrac{1}{T} \right\} = C_Z(t-\tau)$$

Substituting (10) gives:

$$\text{var}\, \tilde{G}(u) = T \int_t \int_\tau g(t) g(\tau) e^{-j2\pi u(t-\tau)} \times \Big[\delta(t-\tau) - \tfrac{1}{T} + \sum_n r_X(t-\tau-nT) - r_X(t-\tau) \Big]$$

$$= TE - |G(u)|^2 + A - B \qquad (11)$$

where

$$E = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-W}^{W} |G(u)|^2 du$$

$$A = T \int_t \int_\tau g(t) g(\tau) e^{-j2\pi u(t-\tau)} \cdot \times \sum_n r_x(t-\tau-nT)\, dt\, d\tau$$

$$B = T \int_t \int_\tau g(t) g(\tau) e^{-j2\pi u(t-\tau)} r_x(t-\tau)\, dt\, d\tau$$

Let's first evaluate A. Using the poisson sum formula:

$$A = \int_t \int_\tau g(t) g(\tau) e^{-j2\pi u(t-\tau)} \times \sum_n |\Phi_x(\tfrac{n}{T})|^2 e^{j2\pi n(t-\tau)/T}\, dt\, d\tau$$

$$= \sum_n |\Phi(\tfrac{n}{T})|^2 \int_t g(t) e^{-j2\pi t(u-\frac{n}{T})} dt \times \int_\tau g(\tau) e^{-j2\pi\tau(\frac{n}{T}-u)} d\tau$$

$$= \sum_n |\Phi(\tfrac{n}{T})|^2 \, G(u-\tfrac{n}{T})\, G(\tfrac{n}{T}-u)$$

Since g is real, $G(u) = G^*(-u)$ and

$$A = \sum_n |\Phi(\tfrac{n}{T})\, G(u-\tfrac{n}{T})|^2 \qquad (12)$$

Now for B: Set $s = t - \tau \Rightarrow \tau = t - s$:

$$B = T \int_t \int_s g(t)\, g(t-s)\, r_X(s)\, e^{-j2\pi u s}\, dt\, ds$$

$$= T \int_s r_g(s)\, r_X(s)\, e^{-j2\pi u s}\, ds$$

where

$$r_g(s) = \int_s g(t)\, g(t-s)\, dt$$

From the convolution theorem:

$$B = T\, |G(u)|^2 * |\Phi_X(u)|^2 \qquad (13)$$

where we have recognized that:

$$r_g(t) \longleftrightarrow |G(u)|^2$$

Substituting (12) & (13) into (11) gives:

$$\operatorname{var} \tilde{G}(u) = TE - |G(u)|^2 + \sum_n \left| \Phi_X\left(\tfrac{n}{T}\right) G\left(u - \tfrac{n}{T}\right) \right|^2 - T\, |G(u)|^2 * |\Phi_X(u)|^2$$

or, since $\Phi_X(0) = 1$,

$$\operatorname{var} \tilde{G}(u) = TE + \sum_{n \neq 0} \left| \Phi\left(\tfrac{n}{T}\right) G\left(u - \tfrac{n}{T}\right) \right|^2 - T\, |G(u)|^2 * |\Phi_X(u)|^2 \qquad (14)$$

(e) If $G(u) = 0$ for $|u| > W$, then,
if $T \leq \frac{1}{2W}$,

$$\sum_{n \neq 0} \left| \Phi\left(\tfrac{n}{T}\right) G\left(u - \tfrac{n}{T}\right) \right|^2 = 0 \quad ; \quad |u| < W$$

and

$$\operatorname{var} \tilde{G}(u) = TE - T\,|G(u)|^2 * |\Phi_X(u)|^2 \qquad (15)$$

For no jitter, $f_X(x) = \delta(x)$ and
$\Phi_X(u) = 1$ and

$$|G(u)|^2 * 1 = \int_{-\infty}^{\infty} |G(u)|^2 du = E$$

and $\operatorname{var} \tilde{G} = 0$ for $|u| < W$

(f) From text, $\operatorname{var} \hat{G}(u) = E/\lambda = ET$ where
$\hat{G}(u)$ is the Poisson estimate. Then, from (15):

$$\operatorname{var} \tilde{G}(u) < \operatorname{var} \hat{G}(u)$$

The jitter estimate is <u>always</u> better,
no matter what f_X or g is.

Extra-Credit.

Great! 100

Gambling with the Odds

Chong Cha
University of Washington

March 11, 1996

Abstract

This paper describes a financial game based on a white sequence of random numbers (stationary discrete stochastic process) to determine the profits (successes) and losses (failures) of a game-player's repeated investments. One free parameter exists which represents the amount the player wishes to reinvestment over all trials in one game. The monetary balance then represents a stochastic process in this free parameter. This paper shows that an optimal reinvestment value can be determined for a given random variable distribution with positive mean (gambling with the odds).

This problem comes from Dr. Robert Marks' class, EE508 (Stochastic Processes) taught at the University of Washington [2]. The three methods of solution include: (1) an analytical method or monotonic manipulation of the equation describing the balance after many trials in a game; (2) a direct calculation for the the optimal reinvestment parameter by numerical integration; and (3) Monte-Carlo simulation.

1 The Game

The game begins with the player's initial I dollars and a choice of a value to reinvest, say f, after each trial in a game (this value remains fixed throughout all N trials per game). The success or failure of each trial is governed by a random variable (RV), C, a set of independent and identically distributed (iid) random numbers (RNs), C_n. Thus, a sequence of trials may follow:

$$\begin{array}{rc}
n=0, & I \\
C_1 > 0: & I(1+P_1 f) \\
C_2 > 0: & I(1+P_1 f)(1+P_2 f) \\
C_3 < 0: & I(1+P_1 f)(1+P_2 f)(1-L_1 f) \\
\vdots & \vdots \\
n=N, & I(1+P_1 f)\ldots(1+P_K f)(1-L_1 f)\ldots(1-L_{N-K} f),
\end{array}$$

for K wins and $N-K$ losses. The final balance for a game (after N total trials) is then

$$\beta(f) = I \prod_{n=1}^{K}(1+P_n f) \prod_{n=1}^{N-K}(1-L_n f), \tag{1}$$

where $P_n = C_n$ for $C_n > 0$ and $L_n = -C_n$ for $C_n < 0$.

For large N,

$$\begin{aligned}\left(\frac{\beta}{I}\right)^{\frac{1}{N}} &\simeq (1+P_1 f)^{p_1}\ldots(1+P_K f)^{p_K} \\ &\quad (1-L_1 f)^{q_1}\ldots(1-L_{N-K} f)^{q_{N-K}},\end{aligned}$$

where $p = \sum_{n=1}^{N} p_n$ is the total probability of success and $q = \sum_{n=1}^{N-K} q_n$ is the total probability of failure. For a uniform distribution of C, all p_n and q_n are equal, but in general,

$$\left(\frac{\beta}{I}\right)^{\frac{1}{N}} = \prod_{n=1}^{N}(1+C_n f)^{Pr(C=C_n)},$$

where C_n can be negative to represent a loss, L_n. Taking the logarithm of both sides gives

$$\begin{aligned}B(f) &\equiv \ln\left(\frac{\beta}{I}\right)^{\frac{1}{N}} \\ &= p_1\ln(1+P_1 f)+\ldots+p_K\ln(1+P_K f)+ \\ &\quad q_1\ln(1-L_1 f)+\ldots+q_{N-K}\ln(1-L_{N-K} f),\end{aligned}$$

which is the expectation of $\ln(1+C_n f)$. Thus the balance can be maximized by maximizing

$$B(f) = E[\ln(1+C_n f)], \tag{2}$$

where the linear operator, $E\cdot$, denotes expectation.

2 Numerical Integration

Equation (2) can be written more explicitly as

$$B(f) = \int_{c=-\infty}^{\infty} \ln(1+cf) f_C(c)\,dc, \tag{3}$$

where $f_C(c)$ is the probability distribution function (pdf) of C.

By numerically integrating equation (3), an optimal value of f can be seen. Figure (1) shows f_{opt} for a uniform distribution. Only for $\eta > 0$ does an optimal reinvestment value exist. Further, a thresholding effect is seen in η, where above

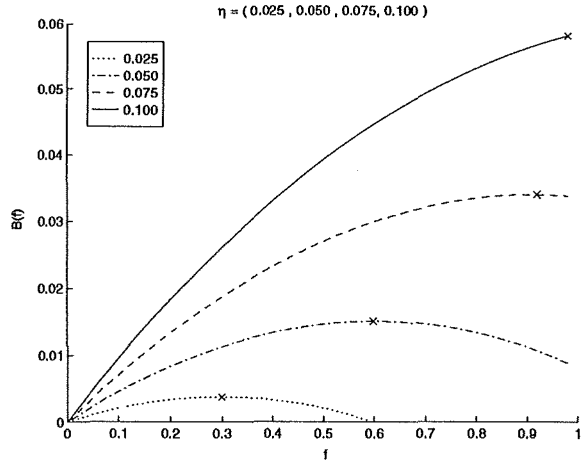


Figure 1: $B(f)$ for C uniformly distributed with various values for the mean, η, of $f_C(c)$. An x represents f_{opt}.

a threshold value of the mean of the pdf, the balance is optimized by reinvesting all of the player's money ($f = 1$). For the unifrom pdf, this value is somewhere between $0.075 < \eta_{thr} < 0.100$.

An optimal f may also be found for a Gaussian (or normal) distribution of C, as seen in figure (2). For a gaussian pdf, the thresholding value of η lies somewhere between $0.225 < \eta_{thr} < 0.300$.

Figure (3) shows the effect on f_{opt} for a larger variance in the normal distribution. Because the values of the profit can take on larger values (due to the larger variance or spread of the pdf in this case as compared to the case with $\sigma = 1$), the thresholding value of the mean is greater: $(\eta_{thr})_{\sigma=1} < (\eta_{thr})_{\sigma=2}$.

3 An Analytical Solution

An explicit equation for the optimal f can be found by analytically integrating equation (3). Assuming a uniform distribution of C with mean η,

$$\begin{aligned}\frac{d}{df}B(f) &= \frac{d}{df}\int_{c=\eta-\frac{1}{2}}^{\eta+\frac{1}{2}} \ln(1+cf)dc \\ &= \frac{f + \ln[1+f(\eta-\frac{1}{2})] - \ln[1+f(\eta+\frac{1}{2})]}{f^2}.\end{aligned}$$

Setting this latter expression to 0 gives $f = f_{opt}$.

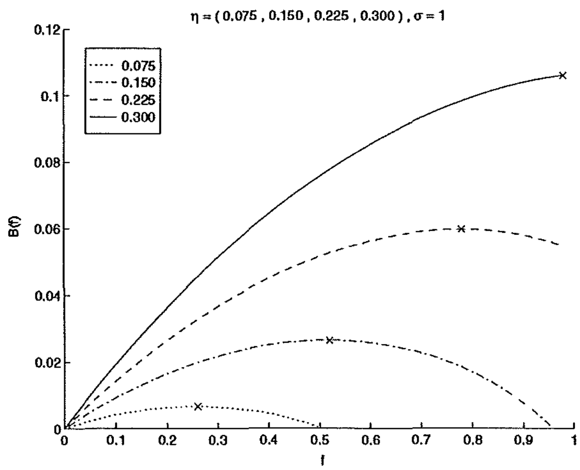


Figure 2: $B(f)$ for C normally distributed with various means and a fixed standard deviation $\sigma = 1$.

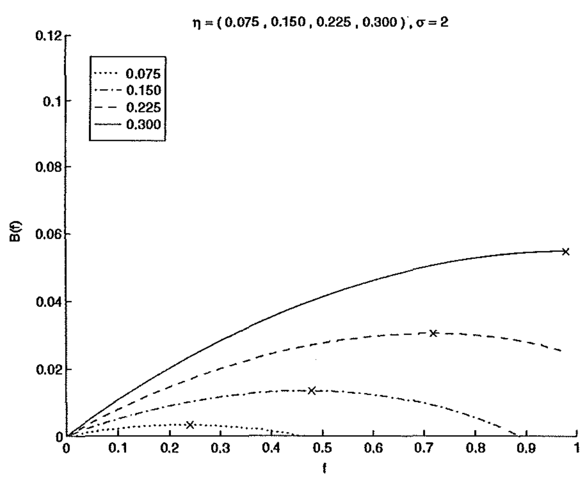


Figure 3: $B(f)$ for C normally distributed with various means and a fixed standard deviation $\sigma = 2$.

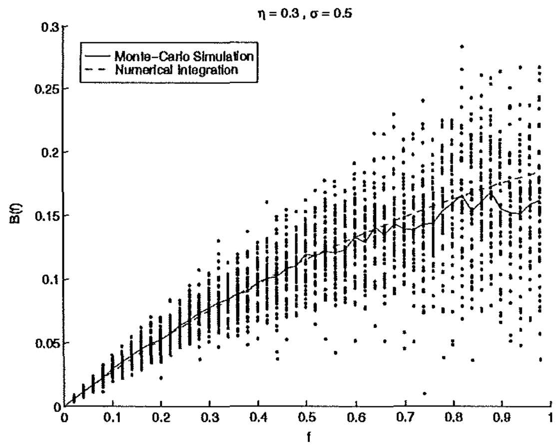


Figure 4: Comparison of $B(f)$ calculated from numerical integration and Monte-Carlo simulation with 50 realisations of $B_{(k)}(f)$. C_n is normally distributed with a mean of $\eta = 0.3$ and a standard deviation $\sigma = 0.5$.

4 Monte Carlo Simulation

For a white sequence of random numbers, one realisation of equation (2) can be written as [1]

$$B_{(k)}(f) \simeq \frac{1}{N}\sum_{n=1}^{N} \ln(1 + C_n f), \tag{4}$$

where the index on $B(f)$ denotes the k^{th} realisation and C_n is the sequence of RNs for a given pdf of C at a particular n.

Using a normal distribution for each C_n, figure (4) shows $B_{(k)}(f)$ for 50 realisations (games) with 100 investments (trials) per game. Also shown, on the same figure, is $B(f)$ calculated from numerical integration (see section 2). The results from the Monte-Carlo simulation seem to corroborate well the solution from the numerical integration method.

It is interesting to consider the effect of varying the variance of the Gaussian distribution on these simulations. Figure (4) showed that for larger investments (larger f), the uncertainty of $B_{(k)}(f)$ goes up (as shown by the trend of more scatter in $B(f)$ for f values nearing unity. Figure (5) shows that the Monte-Carlo simulations are very sensitive to the value of σ: The Monte-Carlo simulations begin to deviate from the correct $B(f)$ curve (taken as that solution obtained from direct integration) at lower reinvestment percentages due to the large uncertainty produced by taking, for example, $\sigma = 1$, while "cleaner" results (less uncertainty) from the true solution are approached for smaller variances.

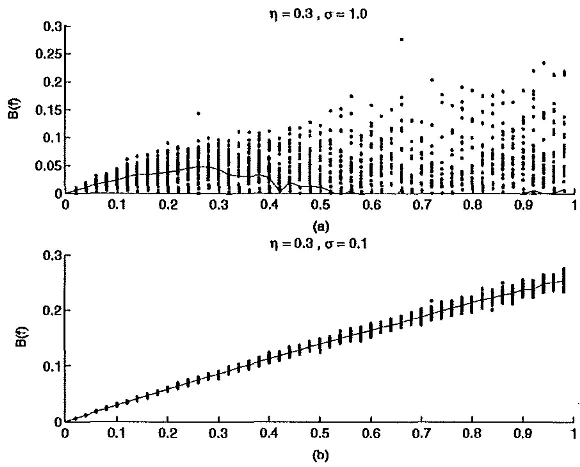


Figure 5: The effect of the variance of C_n on the Monte-Carlo simulation with 50 realisations.

References

[1] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, 1991.

[2] R. J. Marks, *EE508 Lecture Notes*, University of Washington, WiQ 1996.

(2)

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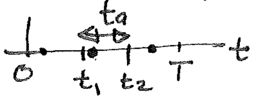
Poisson Review:



n pts @ random

$t_a = t_2 - t_1$; $p = t_a/T$

$P_r[k \text{ PTS on } t_a]$

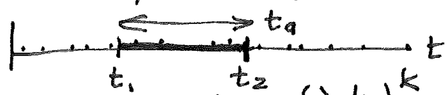
$= \binom{n}{k} p^k q^{n-k}$; $q = 1-p$

$n >> 1$, $p << 1$

$P_r[k \text{ PTS on } t_a] \approx e^{-np} \frac{(np)^k}{k!}$

$= e^{-nt_a/T} \frac{(nt_a/T)^k}{k!}$

Let $n \to \infty$; $T = \infty$; $n/T = \lambda$

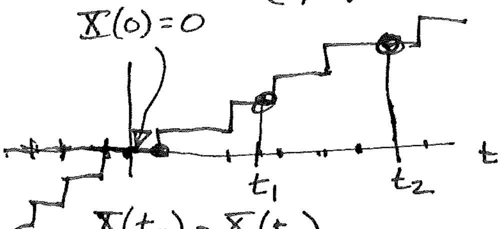


$P_r[k] \Rightarrow e^{-\lambda t_a} \frac{(\lambda t)^k}{k!}$

Poisson Process

$X(t)$ = # PTS on interval $(0,t)$

$X(0) = 0$



$X(t_2) - X(t_1)$

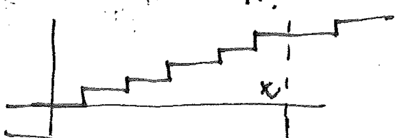
= #PTS on (t_1, t_2)

$\overline{X} = \lambda t \Leftarrow$ claim

@ TIME t

$P_r[k \text{ PTS on } (0,t)]$

$= e^{-\lambda t} \frac{(\lambda t)^k}{k!}$

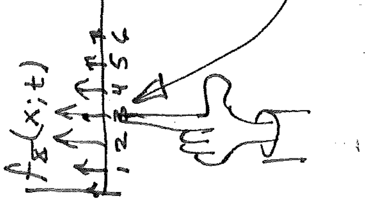


$\overline{X(t)} = \lambda t$

var $X(t)$

$= \lambda t$

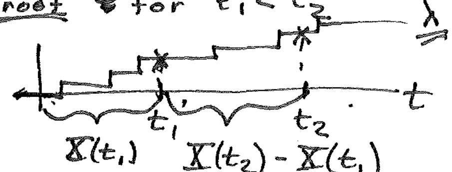
?



Poisson PROCESS

$R(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1,t_2)$

Proof for $t_1 < t_2$



$E[X(t_1)(X(t_2) - X(t_1))]$

$= E[X(t_1)]\, E[X(t_2) - X(t_1)]$

$= \lambda t_1 \cdot \lambda (t_2 - t_1)$

Note:

$\overline{X(t_1) X(t_2)} = R(t_1,t_2)$

$= \overline{X(t_1)[X(t_1) + X(t_2) - X(t_1)]}$

$= \overline{X^2(t_1)} + \overline{X(t_1)(X(t_2) - X(t_1))}$

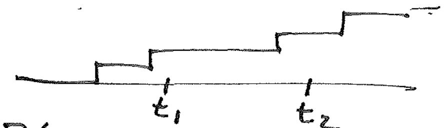
$\overline{X^2(t_1)} = \lambda t_1 + (\lambda t_1)^2$

$= \text{var} X + \overline{X}^2$

$R(t_1,t_2) = \lambda t_1 + (\lambda t_1)^2$

$+ \lambda t_1 \cdot \lambda (t_2 - t_1)$

$= \lambda t_1 + \lambda^2 t_1 t_2$; $t_2 > t_1$



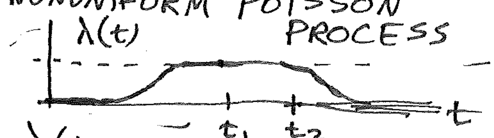
$R(t_1,t_2)$

$= \lambda^2 t_1 t_2 + \lambda \min(t_1,t_2)$

$C(t_1,t_2) = R(t_1,t_2) - \underbrace{\overline{X(t_1)}\,\overline{X(t_2)}}_{\lambda t_1 \lambda t_2}$

$= \lambda \min(t_1,t_2) \Leftarrow$

NONUNIFORM POISSON PROCESS



$\lambda(t_2 - t_1)$

$\Rightarrow \cancel{\int_{t_1}^{t_2} \lambda(\alpha) t}$

$= \int_{t_1}^{t_2} \lambda(\alpha)\, d\alpha$

$\overline{X(t)} = \int_0^t \lambda(\alpha)\, d\alpha$ λt

$R(t_1,t_2)$

TA : Shinhak Lee

email : Sherman@u.wash ———

ROOM : EEB 315

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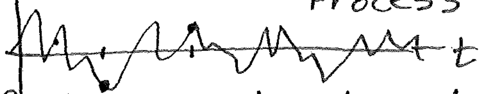
$\underline{X}(t) \leftarrow$ stochastic Process



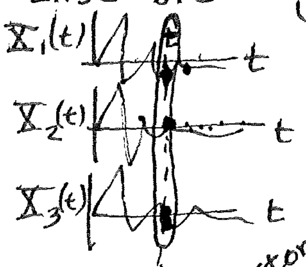
Random Variable depends on time.

$$F(x;t) = P_r[\underline{X} \le x; t]$$

$$F_{\underline{X}(t)}(x;t) = P_r[\underline{X}(t) \le x]$$

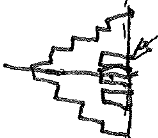
$$f_{\underline{X}(t)}(x;t) = \frac{d}{dx} F_{\underline{X}(t)}(x;t)$$

Ensemble



(First order statistics!

$f_{\underline{X}(t)}(x;t)$



← history

$$E[g(\underline{X}(t))] = \overline{g(\underline{X}(t))}$$

$$= \int_{-\infty}^{\infty} g(x) f_{\underline{X}(t)}(x;t)\,dx$$

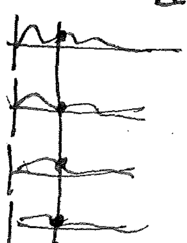
$\overline{\underline{X}(t)}$ = mean

$$\text{ave} = \frac{1}{N}\sum_{n=1}^{N} \underline{X}_n(t) \rightarrow \overline{\underline{X}(t)}$$

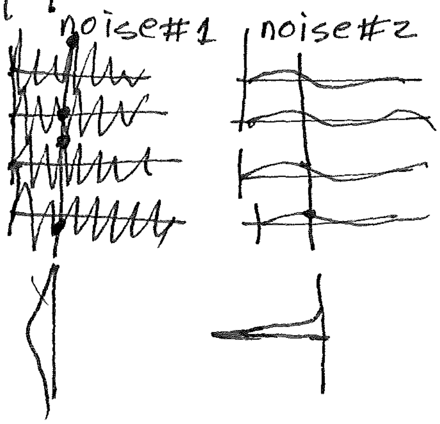
$\text{var}\,\underline{X}(t) =$

$$E\left[\left(\underline{X}(t) - \overline{\underline{X}(t)}\right)^2\right]$$

$$= \overline{\underline{X}^2(t)} - \overline{\underline{X}(t)}^2$$



$\overline{\underline{X}}$



④ A h&w have two children. One is a boy.

$P_r[\text{Other girl}] \neq \frac{1}{2}$

⑤ $\boxed{1}$ $\boxed{2}$ $\boxed{3}$

2 – Garbage

1 – WCCI

① Pick one

② I show you garbage from one of other two.

③ Stay same or change?

① C.O.M. in 2.D
② Uncorr but not ind.
③ Gaussian joint, not marginal

11-22:

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2)$$

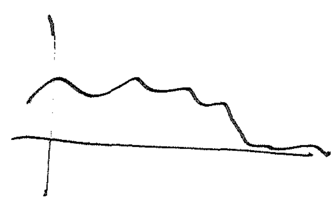
$$t_1 < t_2$$

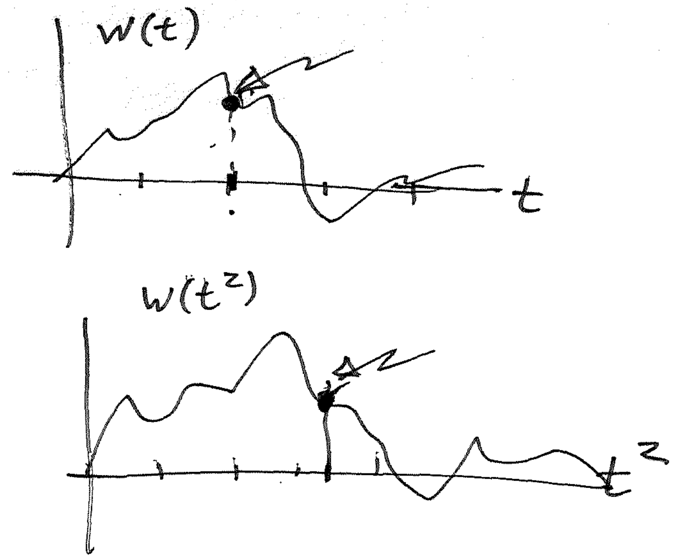
$$R_X(t_1, t_2) = \alpha t_1$$

$$Y(t) = w(t^2)$$

$$R_Y(t_1, t_2) = E\left[w(t_1^2)\, w(t_2^2)\right]$$

$$= \alpha t_1^2$$




W(t)
t
W(t²)
t²

Hint (Eq: 10-95)
See example on p. 138-139

HW: Chapt 10   1, 2, 3, ~~4~~, 5, 8,

\# 3 Chapt 10